Separable Differential Equations

Starter

1. (Review of last lesson) Find the particular solution of the differential equation $\frac{dy}{dx} = e^{2x} - \sin 3x$ given that $x = 0$ when $y = 2$.

Working: $y = \int (e^{2x} - \sin x)dx = \frac{1}{2}e^{2x} + \frac{1}{3}\cos 3x + c$

When $x = 0, y = 2$: $2 = \frac{1}{2} + \frac{1}{3}\cos 0 + c \quad \Rightarrow \quad c = \frac{7}{6}$

$y = \frac{1}{2}e^{2x} + \frac{1}{3}\cos 3x + \frac{7}{6}$

E.g. 1 Find the general solution to the differential equation $\frac{dy}{dx} = \frac{x}{y^2 - 2}$.

Working: $\frac{dy}{dx} = \frac{x}{y^2 - 2}$

Separate the variables: $\int (y^2 - 2)dy = \int x\,dx$

Integrate both sides: $\frac{1}{3}y^3 - 2y = \frac{1}{2}x^2 + c$

Multiply by 6: $2y^3 - 12y = 3x^2 + 6c$

Replace $6c$ by constant A: $2y^3 - 12y = 3x^2 + A$

N.B. Write the constant in the simplest form so A is better than $6c$.

E.g. 2 Which of the following can be solved using the method of separation of variables? Solve those to which the method applies.

(a) $\frac{dy}{dx} = 2x + xy$ (b) $\frac{dy}{dx} = 3x - 2y$

(c) $\frac{dy}{dx} = xy + 6$ (d) $3x + 2y\frac{dy}{dx} = 5$

Working: (a) Can be solved by separating the variables.

$\frac{dy}{dx} = 2x + xy \quad \Rightarrow \quad \frac{dy}{dx} = x(2 + y)$

Separate the variables: $\int \frac{1}{2 + y}dy = \int x\,dx$

Integrate both sides: $\ln|2 + y| = \frac{1}{2}x^2 + c$

$2 + y = e^{\frac{x^2}{2} + c}$

$y = e^{\frac{x^2}{2}} \times e^c - 2$

But e^c is a constant so replace by A: $y = Ae^{\frac{x^2}{2}} - 2$

(b) Cannot be solved by separating the variables.

(c) Cannot be solved by separating the variables.

(d) Can be solved by separating the variables.

$3x + 2y\frac{dy}{dx} = 5 \quad \Rightarrow \quad 2y\frac{dy}{dx} = 5 - 3x$

Separate the variables: $\int 2y\,dy = \int (5 - 3x)\,dx$

Integrate both sides: $y^2 = 5x - \frac{3}{2}x^2 + c$

E.g. 3 Find the particular solution of the differential equation $y^2\frac{dy}{dx} = x^2 + 1$ given that $y = 1$ when $x = 2$

Working: $y^2\frac{dy}{dx} = x^2 + 1 \quad \Rightarrow \quad \int y^2 dy = \int (x^2 + 1)dx$

Integrate both sides: $\frac{1}{3}y^3 = \frac{1}{3}x^3 + x + c$

Multiply by 3: $y^3 = x^3 + 3x + A$

when $x = 2, y = 1$: $1 = 2^3 + 3 \times 2 + A \quad \Rightarrow \quad A = -13$

The particular solution is $y^3 = x^3 + 3x - 13$

E.g. 4 Find the general solution of the differential equation $\frac{dy}{dx} = y - y\cos x$.

Working: $\frac{dy}{dx} = y - y\cos x \quad \Rightarrow \quad \frac{dy}{dx} = y(1 - \cos x)$

Separate the variables: $\int \frac{1}{y}dy = \int (1 - \cos x)dx$

Integrate both sides: $\ln y = x - \sin x + c$

$y = e^{x - \sin x + c} = e^{x - \sin x} \times e^c$

$y = Ae^{x - sinx}$

E.g. 5 Find the particular solution of the differential equation $\frac{dy}{dx} = \sin x\cos^2 y$ given that the curve passes through $\left(\frac{\pi}{2}, \frac{\pi}{4}\right)$.

Working: $\frac{dy}{dx} = \sin x\cos^2 y \quad \Rightarrow \quad \int \frac{1}{\cos^2 y}dy = \int \sin x\,dx$

$\int \sec^2 y\,dy = \int \sin x\,dx$

Integrate both sides: $\tan y = -\cos x + c$

Substitute $\left(\frac{\pi}{2}, \frac{\pi}{4}\right)$***:*** $\tan\frac{\pi}{4} = -\cos\frac{\pi}{2} + c \quad \Rightarrow \quad c = 1$

$\therefore \tan y = 1 - \cos x$

E.g. 6* If $e^{-x^2}\dfrac{dy}{dx} = x(y+2)^2$, find y in terms of x, given that $y=0$ when $x=0$.

Working: $e^{-x^2}\dfrac{dy}{dx} = x(y+2)^2 \quad \Rightarrow \quad \displaystyle\int \frac{1}{(y+2)^2}dy = \int \frac{x}{e^{-x^2}}dx$

$$\int (y+2)^{-2}dy = \int xe^{x^2}dx$$

$$-(y+2)^{-1} = \frac{1}{2}e^{x^2} + c$$

$$-\frac{1}{y+2} = \frac{1}{2}e^{x^2} + c$$

When $x=0, y=0$: $\quad -\dfrac{1}{2} = \dfrac{1}{2} + c \quad \Rightarrow \quad c = -1$

$-\dfrac{1}{y+2} = \dfrac{1}{2}e^{x^2} - 1 \quad \Rightarrow \quad -\dfrac{2}{y+2} = e^{x^2} - 2$

Cross-multiply and expand: $\quad -2 = ye^{x^2} + 2e^{x^2} - 2y - 4$

$$2 - 2e^{x^2} = y(e^{x^2} - 2)$$

$y = \dfrac{2-2e^{x^2}}{e^{x^2}-2} \quad$ or $\quad y = \dfrac{2e^{x^2}-2}{2-e^{x^2}}$

Video: Separation of variables
Video: Separation of variables (ln)
Video: Separation of variables (exp/trig)

Solutions to Starter and E.g.s

Exercise
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