Sigma Notation

Starter

1. **(Review of last lesson)**

A sequence $u_1, u_2, u_3, \ldots$ is defined by $u_1 = 3$ and $u_{n+1} = 1 - \dfrac{1}{u_n}$ for $n \geq 1$.

(a) White down the values of u_2, u_3 and u_4.
(b) Describe the behaviour of the sequence.

Working: (a)
$$u_2 = 1 - \frac{1}{u_1} = 1 - \frac{1}{3} = \frac{2}{3}$$
$$u_3 = 1 - \frac{1}{u_2} = 1 - \frac{1}{\frac{2}{3}} = -\frac{1}{2}$$
$$u_4 = 1 - \frac{1}{u_3} = 1 - \frac{1}{-\frac{1}{2}} = 3$$

(b) The sequence is periodic (or cyclic, or repeating).

2. **(Review of last lesson)** A farmer decides to plant a number of tree as a boundary to a field. The trees are expected to grow by 75 cm per year, so he cuts 20% off their height at the start of each year.

(a) Express this as a linear recurrence relation.
(b) What is the maximum height the trees should grow to?
(c) What percentage of tree height should be cut off each year if he wishes the tree to not exceed 2.5 m?

Working: (a) $u_{n+1} = 0.8u_n + 0.75$ if in metres or $u_{n+1} = 0.8u_n + 75$ if in cm

(b) Maximum height is the limit of the sequence i.e. $L = u_{n+1} = u_n$
$L = 0.8L + 0.75 \quad \Rightarrow \quad 0.2L = 0.75 \quad \Rightarrow \quad L = 3.75$
The maximum height should be 3.75 m

(c) Let the recurrence formula be $u_{n+1} = ku_n + 0.75$
$L = 2.5 = u_{n+1} = u_n$: $\quad 2.5 = 2.5k + 0.75$
$\quad k = 0.7$
The percentage of tree cur should be 30%.

3. A sequence has nth term is $an^2 + b$, where a and b are constants.

(a) If the 3rd term is 7 and the 5th term is 23, find the values of a and b.
(b) Is 35 in the sequence?

Working: (a) 3rd term is 7: $a \times 3^2 + b = 7 \quad \Rightarrow \quad 9a + b = 7$
5th term is 23: $a \times 5^2 + b = 23 \quad \Rightarrow \quad 25a + b = 23$
Solving simultaneously gives $a = 1, b = -2$

(b) nth term is $n^2 - 2$
If 35 is in the sequence: $n^2 - 2 = 35 \quad \Rightarrow \quad n = \sqrt{37}$
Since $n = \sqrt{37}$ is not an integer, 35 is not in the sequence.

E.g. 1 Write out the individual terms of the series and hence evaluate $\sum_{r=3}^{7}(2r-1)$.

Working: $$\sum_{r=3}^{7}(2r-1) = (2\times 3-1)+(2\times 4-1)+(2\times 5-1)+(2\times 5-1)+(2\times 7-1)$$
$$= 5+7+9+11+13 = 45$$

E.g. 2 By writing the series $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots+\frac{1}{50}$ in at least 2 sigma notation ways show that there is not a unique way of expressing a series in sigma notation.

Working: $$\sum_{r=2}^{50}\frac{1}{r} \equiv \sum_{r=1}^{49}\frac{1}{r+1}$$

E.g. 3 Write out the individual terms of these series and hence find their value:

(a) $\sum_{r=0}^{5} r(r+1)$ (b) $\sum_{r=1}^{4}(-1)^r(r+3)$

Working: (a) $$\sum_{r=0}^{5} r(r+1) = 0+1(1+1)+2(2+1)+3(3+1)+4(4+1)+5(5+1)$$
$$= 0+2+6+12+20+30 = 70$$

(b) $$\sum_{r=1}^{4}(-1)^r(r+3) = (-1)^1(1+3)+(-1)^2(2+3)+(-1)^3(3+3)+(-1)^4(4+3)$$
$$= -4+5-6+7 = 2$$

E.g. 4 Express these series in sigma notation

(a) 28 + 22 + 16 + 10 + …. (b) 2 — 4 + 6 — … — 20

Working: (a) Term-to-term rule is —6 ⇒ $-6r$

Term before the first term is $28+6=34$ ⇒ $-6r+34$

$$28+22+16+10+\ldots. = \sum_{r=1}^{\infty}(34-6r)$$

(b) Term-to-term rule is 2 ⇒ $2r$

Term before the first term is $2-2=0$ ⇒ $2r$

Signs go + — + —… so odd terms are positive ⇒ $(-1)^{r+1}$

$$2-4+6-\ldots-20 = \sum_{r=1}^{10}(-1)^{r+1}2r$$

Video: Sigma notation

Solutions to Starter and E.g.s

Exercise

p69 4B Qu 1i, 2i, 3-6