Small angle approximations

Starter

1. Prove the identity $\dfrac{\sin A}{\sin B} + \dfrac{\cos A}{\cos B} \equiv \dfrac{2\sin(A+B)}{\sin 2B}$.

Working:

$$\begin{aligned}\frac{\sin A}{\sin B} + \frac{\cos A}{\cos B} &\equiv \frac{\sin A\cos B}{\sin B\cos B} + \frac{\cos A\sin B}{\sin B\cos B}\\ &\equiv \frac{\sin A\cos B + \cos A\sin B}{\sin B\cos B}\\ &\equiv \frac{\sin(A+B)}{\sin B\cos B}\\ &\equiv \frac{2\sin(A+B)}{2\sin B\cos B}\\ &\equiv \frac{2\sin(A+B)}{2\sin 2B}\end{aligned}$$

E.g. 1 (a) Find the expansion of $\sqrt{1-\theta^2}$ up to and including the term in θ^6.

(b) When θ is small, powers above 2 can be ignored. Write down the approximation for $\cos\theta$.

Working: (a)

$$\begin{aligned}\sqrt{1-\theta^2} &= (1-\theta^2)^{\frac{1}{2}}\\ &= 1 + \frac{1}{2}(-\theta^2) + \frac{\frac{1}{2}\times-\frac{1}{2}}{2!}(-\theta^2)^2 + \frac{\frac{1}{2}\times-\frac{1}{2}\times-\frac{3}{2}}{3!}(-\theta^2)^3 + \ldots\\ &= 1 - \frac{1}{2}\theta^2 - \frac{1}{8}\theta^4 - \frac{1}{16}\theta^6 + \ldots\end{aligned}$$

(b) The small angle approximation for $\cos\theta$ is $1 - \dfrac{1}{2}\theta^2$

E.g. 2 Find approximations for the following when θ is small:

(a) $\dfrac{1-\cos\theta}{\theta^2}$ (b) $\dfrac{\theta\sin\theta}{1-\cos 2\theta}$ (c) $\dfrac{21 + 7\tan\theta - 20\cos\theta}{1+\sin 2\theta}$

Working:

(a) $$\frac{1-\cos\theta}{\theta^2} \approx \frac{1-(1-\frac{1}{2}\theta^2)}{\theta^2} = \frac{\frac{1}{2}\theta^2}{\theta^2} = \frac{1}{2}$$

(b) $$\frac{\theta\sin\theta}{1-\cos 2\theta} \approx \frac{\theta\times\theta}{1-(1-2\theta^2)} \approx \frac{\theta^2}{2\theta^2} = \frac{1}{2}$$

(c) $$\begin{aligned}\frac{21 + 7\tan\theta - 20\cos\theta}{1+\sin 2\theta} &\approx \frac{21 + 7\theta - 20(1-\frac{1}{2}\theta^2)}{1+2\theta}\\ &= \frac{1 + 7\theta + 10\theta^2}{1+2\theta}\\ &= \frac{(1+2\theta)(1+5\theta)}{1+2\theta}\\ &= 1 + 5\theta\end{aligned}$$

E.g. 3 Find: (a) $\lim_{\theta\to 0}\frac{\sin 2\theta}{\theta}$ (b) $\lim_{\theta\to 0}\frac{\tan 7\theta}{\sin\theta}$ (c) $\lim_{\theta\to 0}\frac{\cos 2\theta - 1}{\theta\sin 5\theta}$

Working: (a) $\lim_{\theta\to 0}\frac{\sin 2\theta}{\theta} = \frac{2\theta}{\theta} = 2$

(b) $\lim_{\theta\to 0}\frac{\tan 7\theta}{\sin\theta} = \frac{7\theta}{\theta} = 7$

(c) $\lim_{\theta\to 0}\frac{\cos 2\theta - 1}{\theta\sin 5\theta} = \frac{1-\frac{1}{2}(2\theta)^2 - 1}{\theta\times 5\theta} = \frac{-2\theta^2}{5\theta^2} = -\frac{2}{5}$

E.g. 4 (a) Differentiate $f(x) = \sin x$ from first principles.
(b) Differentiate $f(x) = \cos x$ from first principles.

Hint: the formula for differentiation from first principles is $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{x+h-x}$

Working: (a)

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{x+h-x}\\
&= \lim_{h\to 0}\frac{\sin x\cos h + \cos x\sin h - \sin x}{h} \quad \textit{compound identity}\\
&= \lim_{h\to 0}\frac{\sin x\cos h}{h} + \lim_{h\to 0}\frac{\cos x\sin h}{h} - \lim_{h\to 0}\frac{\sin x}{h} \quad \textit{separate terms}\\
&= \lim_{h\to 0}\frac{\sin x\times\left(1-\frac{1}{2}h^2\right)}{h} + \lim_{h\to 0}\frac{\cos x\times h}{h} - \lim_{h\to 0}\frac{\sin x}{h}\\
&= \lim_{h\to 0}\frac{\sin x}{h} - \lim_{h\to 0}\frac{\frac{1}{2}h^2\sin x}{h} + \lim_{h\to 0}\frac{\cos x\times h}{h} - \lim_{h\to 0}\frac{\sin x}{h}\\
&= -\sin x\times\lim_{h\to 0}\frac{\frac{1}{2}h^2}{h} + \cos x\times\lim_{h\to 0}\frac{h}{h}\\
&= -\sin x\times\lim_{h\to 0}\frac{1}{2}h + \cos x\times 1\\
&= -\sin x\times 0 + \cos x\times 1\\
&= \cos x
\end{aligned}
$$

(b) $f'(x) = \lim_{h\to 0} \dfrac{\cos(x+h) - \cos x}{x+h-x}$

$= \lim_{h\to 0} \dfrac{\cos x \cos h - \sin x \sin h - \cos x}{h}$ ***compound identity***

$= \lim_{h\to 0} \dfrac{\cos x \cos h}{h} - \lim_{h\to 0} \dfrac{\sin x \sin h}{h} - \lim_{h\to 0} \dfrac{\cos x}{h}$ ***separate terms***

$= \lim_{h\to 0} \dfrac{\cos x \times \left(1 - \frac{1}{2}h^2\right)}{h} + \lim_{h\to 0} \dfrac{\sin x \times h}{h} - \lim_{h\to 0} \dfrac{\cos x}{h}$

$= \lim_{h\to 0} \dfrac{\cos x}{h} - \lim_{h\to 0} \dfrac{\frac{1}{2}h^2 \cos x}{h} - \lim_{h\to 0} \dfrac{\sin x \times h}{h} - \lim_{h\to 0} \dfrac{\cos x}{h}$

$= -\cos x \times \lim_{h\to 0} \dfrac{\frac{1}{2}h^2}{h} - \sin x \times \lim_{h\to 0} \dfrac{h}{h}$

$= -\cos x \times \lim_{h\to 0} \dfrac{1}{2}h - \sin x \times 1$

$= -\cos x \times 0 - \sin x \times 1$

$= -\sin x$

E.g. 5 Find an approximation for the expansion $\dfrac{1+\sin\theta}{5+3\tan\theta-4\cos\theta}$ when θ is small i.e. ignore all terms beyond θ^2.

Working:

$$\frac{1+\sin\theta}{5+3\tan\theta-4\cos\theta} \approx \frac{1+\theta}{5+3\theta-4\left(1-\frac{1}{2}\theta^2\right)}$$

$$= \frac{1+\theta}{1+3\theta+2\theta^2}$$

$$= \frac{1+\theta}{(1+\theta)(1+2\theta)}$$

$$= \frac{1}{1+2\theta}$$

$$= (1+2\theta)^{-1}$$

$$= 1 + (-1)\times 2\theta + \frac{-1\times -2}{2!}(2\theta)^2 + \ldots$$

$$\approx 1 - 2\theta + 4\theta^2$$

Video: Small angle approximations

Solutions to Starter and E.g.s

Exercise
p158 7F Qu 1i, 2i, 3i, 4-6, 8-11, (12-13 red)