Solving Geometrical Problems

Starter

1. (Review of last lesson) Decide, with clear working, whether the points $A(1, 0, 3)$, $B(3, 1, 2)$ and $C(7, 3, 0)$ are collinear.

Working: $\overrightarrow{AB} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$ $\qquad \overrightarrow{BC} = 4\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$

$\overrightarrow{BC} = 4\mathbf{i} + 2\mathbf{j} - 2\mathbf{k} = 2 \times (2\mathbf{i} + \mathbf{j} - \mathbf{k}) = 2 \times \overrightarrow{AB}$

Since $\overrightarrow{AB}$ is a multiple of $\overrightarrow{BC}$, the vectors must be parallel.

Since $\overrightarrow{AB}$ and $\overrightarrow{BC}$ have B as a common point and are parallel, it means that A, B and C are collinear.

2. (Review of last lesson) P is the point $(2, -1, 4)$ and Q is the point $(q-2, 5, 2q+1)$. Given that the length of PQ is 11, find the possible coordinates of Q.

Working: $\overrightarrow{PQ} = (q-4)\mathbf{i} + 6\mathbf{j} + (2q-3)\mathbf{k}$

$\sqrt{(q-4)^2 + 6^2 + (2q-3)^2} = 11$

$q^2 - 8q + 16 + 36 + 4q^2 - 12q + 9 = 121$

$5q^2 - 20q - 60 = 0 \quad \Rightarrow \quad q^2 - 4q - 12 = 0$

$\therefore q = 6$ or $q = -2$

When $q = 6$, Q is $(4, 5, 13)$

When $q = -2$, Q is $(-4, 5, -3)$

E.g. 1 Find the unit vector in the direction of $4\mathbf{i} - 4\mathbf{j} - 7\mathbf{k}$.

Working: $|4\mathbf{i} - 4\mathbf{j} - 7\mathbf{k}| = \sqrt{4^2 + (-4)^2 + (-7)^2} = 9$

So the unit vector is $\frac{4}{9}\mathbf{i} - \frac{4}{9}\mathbf{j} - \frac{7}{9}\mathbf{k}$

E.g. 2 Find a vector of magnitude 12 units that is parallel to $-\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$.
Hint: find the unit vector then…

Working: $|-\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}| = \sqrt{(-1)^2 + 2^2 + (-2)^2} = 3$

The unit vector is the direction of $-\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ is $-\frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$

Vector of magnitude 12 $= 12 \times \left(-\frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}\right) = -4\mathbf{i} + 8\mathbf{j} - 8\mathbf{k}$

E.g. 3 Given that $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$, $\mathbf{b} = -\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$ and $\mathbf{c} = -6\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$, find the value of λ such that $\lambda\mathbf{a} + \mathbf{b}$ is parallel to $\mathbf{c}$.

Working: $\lambda\mathbf{a} + \mathbf{b}$ must be a multiple of $\mathbf{c} = -6\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$

i.e. $\lambda\mathbf{a} + \mathbf{b} = k(-6\mathbf{i} + 4\mathbf{j} + 10\mathbf{k})$ where k is a constant

$\lambda\mathbf{a} + \mathbf{b} = \lambda(2\mathbf{i} - 3\mathbf{j} + \mathbf{k}) + -\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$

$= (2\lambda - 1)\mathbf{i} + (4 - 3\lambda)\mathbf{j} + (\lambda - 5)\mathbf{k}$

$(2\lambda - 1)\mathbf{i} + (4 - 3\lambda)\mathbf{j} + (\lambda - 5)\mathbf{k} = k(-6\mathbf{i} + 4\mathbf{j} + 6\mathbf{k})$

Equating components:

$\mathbf{i}$: $2\lambda - 1 = -6k$

$\mathbf{j}$: $4 - 3\lambda = 4k$

$\mathbf{k}$: $\lambda - 5 = 6k$

Solving the $\mathbf{i}$ and $\mathbf{j}$ equations simultaneously gives $\lambda = 2$ and $k = -\frac{1}{2}$

These values work in the $\mathbf{k}$ equation so $\lambda = 2$

E.g. 4 M is the midpoint of the line CD, where C $(-1, 3, -5)$ and $\overrightarrow{CD} = \begin{pmatrix} 4 \\ -4 \\ 6 \end{pmatrix}$. Find the coordinates of M.

Working: Since M is the midpoint: $\frac{1}{2}\overrightarrow{CD} = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$

Then M $= \begin{pmatrix} -1 \\ 3 \\ -5 \end{pmatrix} + \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

So M has coordinates $(1, 1, -2)$

Video A: Vector geometry
Video B: Vector geometry

Solutions to Starter and E.g.s

Exercise
p446 19E Qu 1i, 2-10