The Newton-Raphson Method

Starter

1. (Review of last lesson)

For the equation $x^2 + 2^x - 9 = 0$ show that $x = 2.14$ is a solution to 2 d.p..

Working: Lower bound of 2.14 is 2.135: $f(2.135) < 0$
Upper bound of 2.14 is 2.145: $f(2.145) > 0$
Since there is a sign change between the lower and upper bounds of 2.14, the root of 2.14 is correct to 2 decimal places.

E.g. 1 Use the Newton-Raphson method with a starting point of $x_0 = -2$ to find a root of the equation $f(x) = x^3 - 3x + 5$ correct to 5 decimal places.

Working: ***Find the derivative of the function:*** $f'(x) = 3x^2 - 3$

Substitute the function and its derivative into the formula:

$$x_{n+1} = x_n - \frac{x_n^3 - 3x_n + 5}{3x_n^2 - 3}$$ ***remember to replace x by x_n in the formula***

$x_0 = -2$

Substitute $x_0 = -2$ into formula: $x_1 = -2 - \frac{(-2)^3 - 3\times(-2) + 5}{3\times(-2)^2 - 3} = -\frac{7}{3}$

N.B. Make sure you use brackets around negative numbers in your calculation

Use $x_1 = -\frac{7}{3}$: $x_2 = -\frac{7}{3} - \frac{\left(-\frac{7}{3}\right)^3 - 3\times\left(-\frac{7}{3}\right) + 5}{3\times\left(-\frac{7}{3}\right)^2 - 3} = -2.2080555...$

N.B. Use the back arrow on your calculator so that you don't have to type out the whole calculation again. Then use the ANS button to input the new value into the formula (see the video below for details).

Substitute $x_2 = -2.2080555...$: $x_3 = ... = -2.279020...$
Substitute $x_3 = -2.279020...$: $x_4 = ... = -2.279018...$
Substitute $x_4 = -2.279018...$: $x_4 = ... = -2.279018...$
Since $x_4 = x_5$, this is the value of the root we require.
So $x = -2.27902$ (5 d.p.)

Video: How to use your calculator with Newton-Raphson calculations

E.g. 2 Write down the the Newton-Raphson iterative formula for finding the roots of $f(x) = 5x^2 - 6$.

Working: $f'(x) = 10x$

$$x_{n+1} = x_n - \frac{5x_n^2 - 6}{10x_n}$$ — remember to replace x by x_n in the formula

E.g. 3 By using a starting value of $x_0 = 2.5$, use the Newton-Raphson method to find a solution of the equation $x^4 - 2x^3 = 5$ to 5 s.f.

Working: *Rearrange the equation so that it equals 0:* $x^4 - 2x^3 - 5 = 0$

Form a function: $f(x) = x^4 - 2x^3 - 5$

Find the derivative of the function: $f'(x) = 4x^3 - 6x^2$

Substitute into the formula: $x_{n+1} = x_n - \dfrac{x_n^4 - 2x_n^3 - 5}{4x_n^3 - 6x_n^2}$

N.B. Remember to replace x by x_n in the formula

Substitute $x_0 = 2.5$ into formula: $x_1 = 2.5 - \dfrac{2.5^4 - 2 \times 2.5^3 - 5}{4 \times 2.5^3 - 6 \times 2.5^2}$

$x_1 = 2.3875$

N.B. Use the back button and replace 2.5 by ANS in the formula

Substitute x_1 into the formula: $x_2 = 2.37398...$

N.B. Now you can just press the '=' button to get subsequent iterations

Substitute x_2 into the formula: $x_3 = 2.37380...$

Substitute x_3 into the formula: $x_4 = 2.37380...$

Since $x_3 = x_4$, this is the value of the root we require.

So $x = 2.3738$ (5 s.f.)

E.g. 4 Solve the equation $x^3 + 3x^2 + 5x + 7 = 0$ using the Newton-Raphson method and starting with $x_0 = 1$. Give your answer to 4 s.f.

Working: *Form a function:* $f(x) = x^3 + 3x^2 + 5x + 7$

Find the derivative of the function: $f'(x) = 3x^2 + 6x + 5$

Substitute into the formula: $x_{n+1} = x_n - \dfrac{x_n^3 + 3x_n^2 + 5x_n + 7}{3x_n^2 + 6x_n + 5}$

Substitute $x_0 = 1$ into formula: $x_1 = 1 - \dfrac{1^3 + 3 \times 1^2 + 5 \times 1 + 7}{3 \times 1^2 + 6 \times 1 + 5}$

$x_1 = -\dfrac{1}{7}$

Substitute x_1 into the formula: $x_2 = -1.6519...$

Substitute x_2 into the formula: $x_3 = -2.3906$

Substitute x_3 into the formula: $x_4 = -2.2021$

Substitute x_4 into the formula: $x_5 = -2.1798$

Substitute x_5 into the formula: $x_6 = -2.1795$

Substitute x_6 into the formula: $x_7 = -2.1795$

Since $x_6 = x_7$, this is the value of the root we require.

So $x = -2.180$ (4 s.f.)

Video: Newton-Raphson method

Video: How to use your calculator with Newton-Raphson calculations

Newton-Raphson method EQ

Solutions to Starter and E.g.s

Exercise

Beware of examples involving special functions.

p307 14B 1a (i & ii), 2 (not bi), 3, 4, 6, 7

Hints: Qn 2bii: the derivative of e^{kx} is ke^{kx}

Qu 6: multiply the equation by x before finding the Newton-Raphson formula