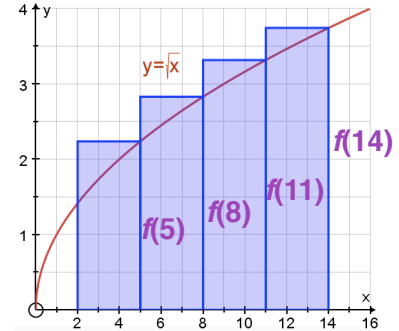
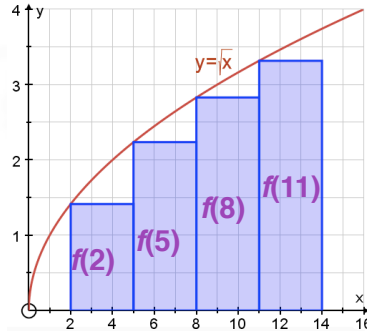
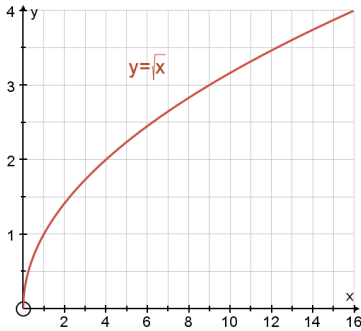


## Upper and Lower Bounds of Integration

**E.g. 1** Find upper and lower bounds for the integration  $\int_2^{14} \sqrt{x} dx$ , using 4 rectangles, giving each answer to 3 s.f.

**Working:**



4 rectangles to fit between  $x = 2$  and  $x = 14$

So the width of each rectangle is  $\frac{14 - 2}{4} = 3$

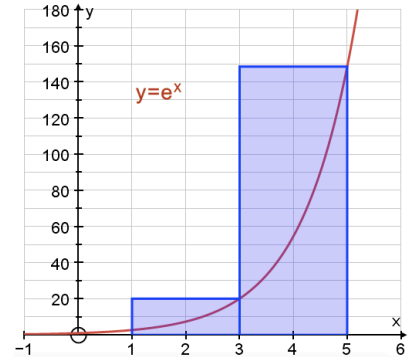
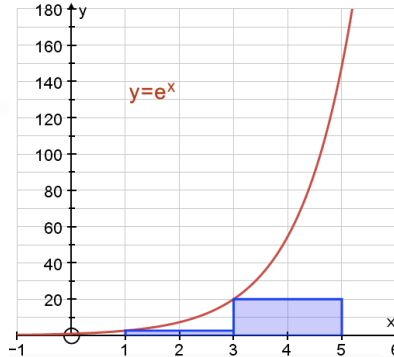
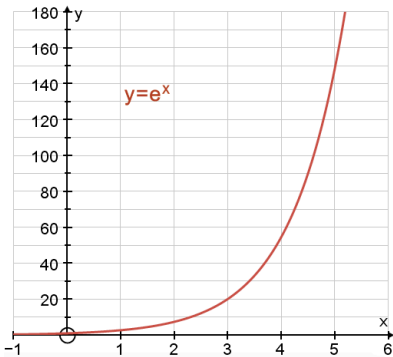
$$\begin{aligned} \text{Lower bound} &= 3 \times f(2) + 3 \times f(5) + 3 \times f(8) + 3 \times f(11) \\ &= 3 \times \sqrt{2} + 3 \times \sqrt{5} + 3 \times \sqrt{8} + 3 \times \sqrt{11} \\ &= 29.4 \text{ (3 s.f.)} \end{aligned}$$

$$\begin{aligned} \text{Upper bound} &= 3 \times f(5) + 3 \times f(8) + 3 \times f(11) + 3 \times f(14) \\ &= 3 \times \sqrt{5} + 3 \times \sqrt{8} + 3 \times \sqrt{11} + 3 \times \sqrt{14} \\ &= 36.4 \text{ (3 s.f.)} \end{aligned}$$

$$\text{So } 29.4 < \int_2^{14} \sqrt{x} dx < 36.4$$

**E.g. 2** Use two rectangles to give upper and lower bounds for  $\int_1^5 e^x dx$ , giving each answer to 3 s.f..

**Working:**



2 rectangles to fit between  $x = 1$  and  $x = 5$

So the width of each rectangle is  $\frac{5-1}{2} = 2$

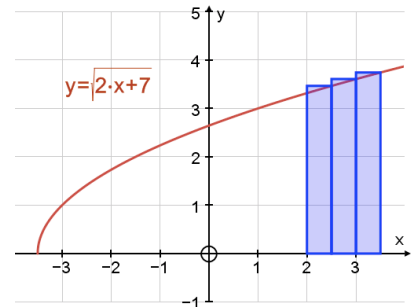
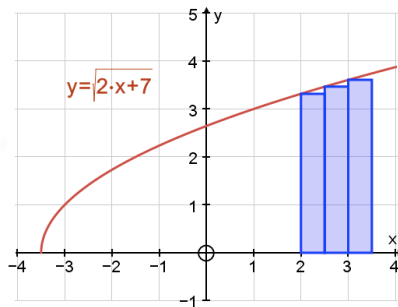
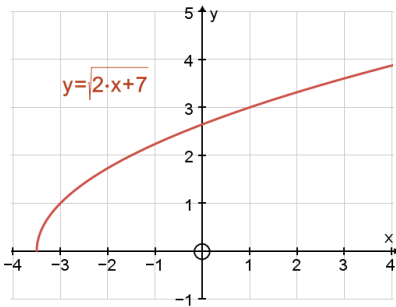
$$\begin{aligned} \text{Lower bound} &= 2 \times f(1) + 2 \times f(3) \\ &= 2 \times e^1 + 2 \times e^3 \\ &= 45.6 \text{ (3 s.f.)} \end{aligned}$$

$$\begin{aligned} \text{Upper bound} &= 2 \times f(1) + 2 \times f(3) \\ &= 2 \times e^1 + 2 \times e^3 \\ &= 337 \text{ (3 s.f.)} \end{aligned}$$

So  $45.6 < \int_1^5 e^x dx < 337$  *these estimates are not very useful*

**E.g. 3** Use three rectangles to give upper and lower bounds for  $\int_2^{3.5} \sqrt{2x+7} dx$

**Working:**



3 rectangles to fit between  $x = 2$  and  $x = 3.5$

So the width of each rectangle is  $\frac{3.5 - 2}{3} = 0.5$

$$\begin{aligned} \text{Lower bound} &= 0.5 \times f(2) + 0.5 \times f(2.5) + 0.5 \times f(3) \\ &= 0.5 \times \sqrt{2 \times 2 + 7} + 0.5 \times \sqrt{2 \times 2.5 + 7} + 0.5 \times \sqrt{2 \times 3 + 7} \\ &= 5.19 \text{ (3 s.f.)} \end{aligned}$$

$$\begin{aligned} \text{Upper bound} &= 0.5 \times f(2.5) + 0.5 \times f(3) + 0.5 \times f(3.5) \\ &= 0.5 \times \sqrt{2 \times 2.5 + 7} + 0.5 \times \sqrt{2 \times 3 + 7} + 0.5 \times \sqrt{2 \times 3.5 + 7} \\ &= 5.41 \text{ (3 s.f.)} \end{aligned}$$

$$\text{So } 5.19 < \int_2^{3.5} \sqrt{2x+7} dx < 5.41 \quad \text{good estimates}$$

Video: [Riemann approximation](#)  
 Video: [Numerical integration with rectangles](#)  
[Riemann approximation \(notes\)](#)

[Solutions to Starter and E.g.s](#)

**Exercise**

p334 15A Qu 1aii, 1bi (use radians), 1cii (use radians), 3, 4, 6b