Using Sequences to Solve Problems

Starter

1. **(Review of last lesson)**

The sum to infinity of a convergent geometric series is equal to the sum to infinity of the squares of its terms. If the first term of the geometric series is $\frac{1}{4}$, find its common ratio.

Working:

Geometric series: $\frac{1}{4} + \frac{1}{4}r + \frac{1}{4}r^2 + \frac{1}{4}r^3 + \dots$

Using $S_\infty = \frac{a}{1-r}$: $S_\infty = \frac{0.25}{1-r} = \frac{1}{4-4r}$

Square of terms: $\frac{1}{16} + \frac{1}{16}r^2 + \frac{1}{16}r^4 + \frac{1}{16}r^6 + \dots$

Using $S_\infty = \frac{a}{1-r}$: $S_\infty = \frac{1}{16(1-r^2)} = \frac{1}{16-16r^2}$

The sums to infinity are equal: $\frac{1}{4-4r} = \frac{1}{16-16r^2}$

Cross-multiply: $16 - 16r^2 = 4 - 4r$

Collect like terms: $16r^2 - 4r - 12 = 0$

Divide by 4: $4r^2 - r - 3 = 0$

Factorise to solve: $(4r+3)(r-1) = 0$

$r = -\frac{3}{4}$ or $r = 1$

Since $r \neq 1$ (otherwise all the terms would be the same and the series would not converge), the common ratio is $-\frac{3}{4}$

2. When a baby is born, £3000 is invested in an account with a fixed interest rate of 2.8% per year.

(a) What will the account be worth at the end of the 7th year?

(b) After how many full years will the account have doubled in value?

Working:

(a) $3000 \times 1.028^7 = £3639.76$

(b) $3000 \times 1.028^n = 6000 \quad \Rightarrow \quad 1.028^n = 2$

Take $\ln$ of both sides: $\ln 1.028^n = \ln 2$

3rd law of logs: $n \ln 1.028 = \ln 2$

$$n = \frac{\ln 2}{\ln 1.028} = 25.1$$

After 26 years

E.g. 1 A car depreciates by 15% each year. After 10 years it is worth £2362. How much was it worth when new?

Working: Depreciates by 15% $\Rightarrow \quad \times 0.85$

Let v be the value of the car when new $\Rightarrow \quad v \times 0.85^{10} = 2362$

$$v = \frac{2362}{0.85^{10}} = 11997.50$$

The car cost £11 997.50 new.

E.g. 2 (a) Katie invests £6000 at an annual interest rate of 6%. Find the value of the investment after 10 years.

(b) Olivia invests £6000 on the 1st January each year. Interest of 6% is paid on the whole amount in the bank on 31st December each year. Calculate the value of the investment at the end of 10 years and the amount of interest she has gained.

Working: (a) $6000 \times 1.06^{10} = £10745.09$

(b) 1st year, 1st Jan: 6000

31st Dec: 6000×1.06

2nd year, 1st Jan: $6000 \times 1.06 + 6000$

31st Dec: $(6000 \times 1.06 + 6000) \times 1.06$

$= 6000 \times 1.06^2 + 6000 \times 1.06$

$= 6000(1.06^2 + 1.06)$

10th year, 31st Dec: $6000(1.06^{10} + 1.06^9 + 1.06^8 + \ldots + 1.06^2 + 1.06)$

$a = 6000 \times 1.06, \; r = 1.06$

Since $r > 1$ use $S_n = \dfrac{a(r^n - 1)}{r - 1}$: $\quad S_{10} = \dfrac{6000 \times 1.06(1.06^{10} - 1)}{1.06 - 1}$

$= £83829.86$

Interest gained $= 83829.86 - 60000 = £23829.86$

Video: Investment problems

Geometric progressions EQ 1
Geometric progressions EQ 2

Solutions to Starter and E.g.s

Exercise
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