L6 Further Mathematics January Exam Teacher X 24-25 SOLUTIONS [55]

1.

(a)		$\mathbf{C}=2\begin{pmatrix}4&-3\\-2&2\end{pmatrix}-4\begin{pmatrix}3&-5\\0&1\end{pmatrix}$ $=\begin{pmatrix}8&-6\\-4&4\end{pmatrix}-\begin{pmatrix}12&-20\\0&4\end{pmatrix}$ $=\begin{pmatrix}8-12&-6--20\\-4-0&4-4\end{pmatrix}$ $=\begin{pmatrix}-4&14\\-4&0\end{pmatrix}$	M1 A1 [2]	1.1 1.1	Sufficient working to demonstrate knowledge of scalar multiplication of a matrix and subtraction of matrices. Can be implied by 3 out of 4 entries correct.	Or BC
(b)		$(\mathbf{C}=)\begin{pmatrix}2&0\\0&2\end{pmatrix}$ or $2\mathbf{I}$	B1 [1]	2.2a		
(c)		$(\det\mathbf{A} = 4\times2-(-3)(-2) = 8-6 =)\ 2$	B1 [1]	1.1		
(d)		$\begin{pmatrix}4&-3\\-2&2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}7\\9\end{pmatrix}$ $\frac{1}{2}\begin{pmatrix}2&3\\2&4\end{pmatrix}\begin{pmatrix}7\\9\end{pmatrix}$ or $\mathbf{A}^{-1}\begin{pmatrix}7\\9\end{pmatrix}$ if $\mathbf{A}^{-1}$ defined so $x=\frac{41}{2}$, $y=25$	M1 M1 A1 [3]	1.1 1.1 1.1	**DR** Expressing the system in matrix form. Can be implied by the next line. Forming correct solution as matrix/vector product with inverse matrix. 20½ or 20.5	Matrix method must be used. Any other method 0/3. FT their detA from (c). Except for this, inverse must be correct. Condone $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\frac{41}{2}\\25\end{pmatrix}$ but not $\begin{pmatrix}x\\y\end{pmatrix}=\frac{1}{2}\begin{pmatrix}41\\50\end{pmatrix}$

2.

(a)	**(i)**	$\begin{pmatrix}1\\2\\-1\end{pmatrix}\times\begin{pmatrix}3\\5\\-2\end{pmatrix}=\begin{pmatrix}1\\-1\\-1\end{pmatrix}$	B1 [1]	1.1		
	(ii)	$\begin{pmatrix}1\\-1\\-1\end{pmatrix}$ is perpendicular to [both] $\begin{pmatrix}1\\2\\-1\end{pmatrix}$ and [also] $\begin{pmatrix}3\\5\\-2\end{pmatrix}$	B1FT [1]	1.2	This can be stated as a generality (eg **a**×**b** is perpendicular to both **a** and also **b**). FT their answer to **(a)(i)** (ie the mark can be awarded if the property is stated for their vector which is not, in fact, perpendicular).	Accept "perpendicular to both vectors" or even just "they are perpendicular" But if vectors stated must be correct vectors (or their vectors if MR in part **(i)**). If cross product vector stated must be their cross product vector.
	(iii)	If the dot product is zero then the vectors are perpendicular $\begin{pmatrix}1\\2\\-1\end{pmatrix}.\begin{pmatrix}1\\-1\\-1\end{pmatrix}=1-2+1=0$ $\begin{pmatrix}3\\5\\-2\end{pmatrix}.\begin{pmatrix}1\\-1\\-1\end{pmatrix}=3-5+2=0$ (so the answer to **(a)(i) is** perpendicular to both as claimed.)	M1 A1 [2]	2.1 2.2a	**M1** can be implied by Attempt to find a relevant dot product and show that this is 0 Correct calculation of **a**.(**a**×**b**) and **b**.(**a**×**b**), showing some details of calculation (ie not simply stating "= 0" without justification).	NB Both these marks are still available to candidates whose answer to **(a)(i)** is $\begin{pmatrix}-1\\1\\1\end{pmatrix}$.
(b)		$(2\mathbf{i}-2\mathbf{j}+\mathbf{k}).(4\mathbf{i}-\mathbf{j}+8\mathbf{k}) = 8+2+8 = 18$ $\cos\theta=\frac{a.b}{\lvert a\rvert\lvert b\rvert}$ $=\frac{18}{\sqrt{2^2+(-2)^2+1^2}\sqrt{4^2+(-1)^2+(-8)^2}}$ $=\frac{18}{\sqrt{9}\sqrt{81}}=\frac{2}{3}$ $\therefore\theta=\cos^{-1}\left(\frac{2}{3}\right)=48.2°\ (1\ dp)$	B1 M1 A1 [3]	1.1 1.1 1.1	Can be implied by correct answer Correct method for evaluation of cosine of required angle, including correct form for both moduli. awrt 48.2° or 0.841 rads.	**M1** can be awarded after correct rearrangement of $\mathbf{a.b}=\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\theta$ once correct form for moduli seen even if subsequent (calculation) error. 48.1896851… or 0.8410686706

3.

(a)		$\overrightarrow{AB}=\begin{pmatrix}11\\-9\\0\end{pmatrix}-\begin{pmatrix}8\\-7\\-2\end{pmatrix}=\begin{pmatrix}3\\-2\\2\end{pmatrix}$	**M1**	1.1	Subtracting the position vectors of A and B (in either order). Calculation or answer is enough for M1	Allow M1 for a row vector i.e. (3, -2, 2)
		$\mathbf{r}=\begin{pmatrix}8\\-7\\-2\end{pmatrix}+\lambda\begin{pmatrix}3\\-2\\2\end{pmatrix}$	**A1** **[2]**	1.1	Must be "**r** =" (or "$\begin{pmatrix}x\\y\\z\end{pmatrix}$ =") Allow r or **L**$_1$ but not L$_1$ (i.e. allow r not to be underlined as a vector, but if L$_1$ used this must be indicated as a vector.	$\mathbf{r}=\begin{pmatrix}8\\-7\\-2\end{pmatrix}$ or $\begin{pmatrix}11\\-9\\0\end{pmatrix}\pm\lambda\begin{pmatrix}3\\-2\\2\end{pmatrix}$ Note – could use other "starting points" Must be column vectors here
(b)		$8+3\lambda=26$ or $-7-2\lambda=-19$ or $-2+2\lambda=-14$	**M1**	1.1	Writing down a correct equation for any component of their declared line.	Note – values of λ will depend on the "starting point" used in part **(a)** and different multiples of the direction vector.
		(x gives $\lambda=6$ or y gives or $\lambda=6$) [but] z gives $\lambda=-6$ so $(26, -19, -14)$ does **not** lie on the line	**A1FT** **[2]**	2.2a	Finding a correct inconsistency and reaching correct conclusion. Not all values need be given but values given must be correct. FT their declared line.	Could see eg $\lambda=6$ => ($y=-19$ but) $z=10$ which is not -14 Do not need to see the word "inconsistent" or explicit comparison. Finding two different values of λ and then stating does not lie on line is enough
(b)		**Alternate method** $\begin{pmatrix}18\\-12\\-12\end{pmatrix}=\lambda\begin{pmatrix}3\\-2\\2\end{pmatrix}$	**M1**		Rearranging vector equation	
		But $\begin{pmatrix}3\\-2\\2\end{pmatrix}$ is not parallel to $\begin{pmatrix}18\\-12\\-12\end{pmatrix}$ so the point does not lie on the line	**A1FT**		Or finding λ inconsistency as before	A0 for "there is no value of λ that works" without justification
(c)		$\overrightarrow{OC}=\begin{pmatrix}8\\-7\\-2\end{pmatrix}+\lambda\begin{pmatrix}3\\-2\\2\end{pmatrix}$ [for some particular value of λ.]	**B1FT**	3.1a	Can be embedded or implied. Can be same or different symbol as parameter in **(a)**.	
		[$\overrightarrow{OC}=\mu\begin{pmatrix}a\\b\\c\end{pmatrix}$] and $\begin{pmatrix}a\\b\\c\end{pmatrix}.\begin{pmatrix}3\\-2\\2\end{pmatrix}=0$	**B1FT**	2.1	Condition for perpendicularity stated.	Don't have to see an equation for OC in terms of μ here.
		$3a-2b+2c=0$ $3(8+3\lambda)-2(-7-2\lambda)+2(-2+2\lambda)=0$	**M1**	1.1	Forming a correct dot product and using it with the other definition of $\overrightarrow{OC}$ to form an equation in the parameter.	μ may or may not be present here, but must be dealt with appropriately for the A mark.
		=> $\lambda=-2$ => $\overrightarrow{OC}=\begin{pmatrix}8\\-7\\-2\end{pmatrix}-2\begin{pmatrix}3\\-2\\2\end{pmatrix}=\begin{pmatrix}2\\-3\\-6\end{pmatrix}$ so the equation of L_2 is $\mathbf{r}=\mu\begin{pmatrix}2\\-3\\-6\end{pmatrix}$	**A1** **[4]**	2.2a	Condone use of the same parameter as L_1. Could see eg $\mathbf{r}=\begin{pmatrix}2\\-3\\-6\end{pmatrix}+\mu\begin{pmatrix}2\\-3\\-6\end{pmatrix}$	BOD lack of "**r** =" (or "$\begin{pmatrix}x\\y\\z\end{pmatrix}$ =") in this part if has already been penalised in part (a). If full marks awarded in part (a) then must have correct notation here for full marks.

(c)		**Alternative Method:** $\overrightarrow{OC} = \begin{pmatrix} 8 \\ -7 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$ for some particular value of λ.	B1FT		Can be embedded or implied. Can be same or different symbol as parameter in **(a)**.	
		$\overrightarrow{OC}.\mathbf{b}_{L_1} = 0$	B1FT		Condition for perpendicularity stated.	
		$\therefore \left(\begin{pmatrix} 8 \\ -7 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} \right) . \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$ $= 24 + 14 - 4 + (9 + 4 + 4)\lambda = 0$	M1		Forming a correct dot product and using it to form an equation in the parameter.	$\therefore \begin{pmatrix} 8+3\lambda \\ -7-2\lambda \\ 10+2\lambda \end{pmatrix} . \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$ $= 24 + 9\lambda + 14 + 4\lambda + 20 + 4\lambda = 0$
		$\Rightarrow \lambda = -2 \Rightarrow \overrightarrow{OC} = \begin{pmatrix} 8 \\ -7 \\ -2 \end{pmatrix} - 2 \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ -6 \end{pmatrix}$ so the equation of L_2 is $\mathbf{r} = \mu \begin{pmatrix} 2 \\ -3 \\ -6 \end{pmatrix}$	A1		Could see eg $\mathbf{r} = \begin{pmatrix} 2 \\ -3 \\ -6 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -3 \\ -6 \end{pmatrix}$ Condone use of the same parameter as L_1.	Condone presence of zero vector as first point BOD lack of "$\mathbf{r} =$" (or "$\begin{pmatrix} x \\ y \\ z \end{pmatrix} =$") in this part if has already been penalised in part (a). If full marks awarded in part (a) then must have correct notation here for full marks
(c)		**Alternative Method 2:** $\overrightarrow{OC} = \begin{pmatrix} 8 \\ -7 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$ for some particular value of λ.	B1FT		Can be embedded or implied. Can be same or different symbol as parameter in **(a)**.	

4.

(a)		$\mathbf{A}^{-1} = \frac{1}{3}\begin{pmatrix} 3 & 0 \\ -2 & 1 \end{pmatrix}$	**B1** **[1]**	1.1		
(b)		$[\mathbf{AB} =]\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & -3 \end{pmatrix}$	**M1**	1.1	Correctly finding either **AB** or **BA**.	
		$[\mathbf{BA} =]\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & -3 \end{pmatrix}$ so they are not the same	**A1** **[2]**	2.2a	Must give correct calculation and correct conclusion.	$\mathbf{AB} \neq \mathbf{BA}$ is fine for conclusion. General statement "matrices not commutable not OK unless linked to particular case.
(c)		The matrix representing R is **C**(**BA**)...	**M1**	3.1a	Or the matrix representing S is (**CB**)**A** Must have **R=C(BA)** or **S=(CB)A** with brackets or equivalent correct	Need to see the brackets or equivalent BOD sight of T_A
		...and the matrix representing S is (**CB**)**A** and by associativity (of matrix multiplication), **C**(**BA**) = (**CB**)**A** (so R and S are the same)	**A1** **[2]**	3.2a	Correct form for S and either explicit equality or associativity property explicitly mentioned. M1 A0 only if state that matrices are commutative	If **M0** then **SC B1** for observation that (**AB**)**C** = **A**(**BC**) or any other correct statement of associativity of matrix multiplication.
(d)	**(i)**	det**A** = 3 or det**B** = –1	**M1**	1.1	Correctly calculating the determinant of A or B	Question says "Determine" so answer only is 0
		det (**CBA**) = $-3(a^2 + 1) < 0$ (since a is real) [so the orientation is reversed]. Order is *D'*, *G'*, *F'*, *E'*	**A1** **[2]**	2.2a	Some justification must be given but condone incorrect order of matrices. Orientation reversed is fine, but must see correct det (**CBA**) **SC1** If neither det **A** or det **B** explicitly calculated then allow B1 for det **A**, det **C** positive and det **B** negative so orientation reversed.	Can consider det(**C**) det(**B**) det(**A**) instead, i.e. the effect on orientation of each of the three transformations Allow "the order is clockwise". **SC2** If only det (C) considered (i.e. candidate thinks transformation R is represented by **C**) then allow **B1** for $a^2+1>0$ so orientation is the same
	(ii)	$15(a^2 + 1)$	**B1FT** **[1]**	1.1	FT 5× | their determinant from **(i)** | (if found)	FT only if their determinant is negative

5.

Base case: when $n=0$, $11\times7^0-13^0-1=9$ which is divisible by 3.	B1	2.5	Base case shown to be true – condone when $n = 0$, $11 - 1 - 1 = 9$ or just 9, must say that 9 is divisible by 3 or show explicitly e.g. $9=3\times3$. Allow base case with $n = 1$, $11\times7-13-1=63$ or just 63, must say that 63 is divisible by 3 or show explicitly e.g. $63=3\times21$.
Assume that, when $n=k$, $11\times7^n-13^n-1$ is divisible by 3. That is, $11\times7^k-13^k-1=3m$ (for some integer m).	M1*	2.1	Forms correct inductive hypothesis as an equation in terms of two different letters (oe in words e.g. $11\times7^k-13^k-1$ is divisible by 3) – condone using n e.g. $11\times7^n-13^n-1=3m$ for this mark.
$11\times7^{k+1}-13^{k+1}-1$ $=7\times(3m+13^k+1)-13^{k+1}-1$	M1dep*	3.1a	Considers $11\times7^{k+1}-13^{k+1}-1$ and use inductive hypothesis that $11\times7^k-13^k-1=3m$ - must not be using n for k.
$=3(7m+2-2\times13^k)$ (which is divisible by 3).	A1	2.2a	Shows using the inductive hypothesis that $11\times7^{k+1}-13^{k+1}-1$ is a multiple of 3. Some common answers are $3(-22\times7^k+4+13m)$ or $3(7m+2-2\times13^k)$ or $3(m+22\times7^k-4\times13^k)$.
So, if true for $n = k$ this implies true for $n = k + 1$. True when $n = 0$ so therefore true for all integers $n\geqslant0$.	A1	2.4	Conclusion, dependent on **B1M1M1A1** and no errors seen in their proof. Do not award this mark if $n = 0$ not used as base case. However, this mark can be awarded if $n = 1$ used as base case and $n = 0$ considered separately. Must mention, '$n = k$ **or** P(k)', '$n = k + 1$ **or** P($k + 1$)', '$n = 0$ **or** base case **or** P(0)' (provided P(n) is defined) and '$n\geqslant0$ **or** all n oe' condone 'all integers' but not 'positive integers' or similar incorrect statement.
	[5]		

6.

(a)

Considers a general point (x, y) Multiplies **M** $\begin{bmatrix}x\\y\end{bmatrix}$ to form at least one correct equation. May be unsimplified. Or Shows that the origin is invariant, eg writes **M** $\begin{bmatrix}0\\0\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$ eg states that the origin is always invariant under a linear transformation.	3.1a	M1	$\begin{bmatrix}3&-1\\-2&6\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x\\y\end{bmatrix}$ $3x-y=x$ and $-2x+6y=y$ $y=2x$ and $2x=5y$ $x=0$ and $y=0$ So (0,0) is the only invariant point
Forms two different correct equations in x and y	1.1a	M1	
Completes a reasoned argument and concludes that the origin is the only invariant point.	2.1	R1	
Subtotal		**3**	

(b)

Writes the product $\mathbf{M}\begin{bmatrix} x \\ x+1 \end{bmatrix}$ and equates to $\begin{bmatrix} X \\ mX+c \end{bmatrix}$ or equivalent. PI by one correct equation. **Or** Multiplies **M** by a specific point on the line $y = x + 1$	3.1a	M1	$\begin{bmatrix} 3 & -1 \\ -2 & 6 \end{bmatrix}\begin{bmatrix} x \\ x+1 \end{bmatrix} = \begin{bmatrix} X \\ mX+c \end{bmatrix}$ $3x - 1(x + 1) = X$ and $-2x + 6(x + 1) = mX + c$ $2x - 1 = X$ and $4x + 6 = mX + c$ So $4x + 6 \equiv m(2x - 1) + c$ $4 = 2m$ and $6 = -m + c$ $m = 2$ $6 = -2 + c$ $c = 8$ The new line is $y = 2x + 8$
Obtains two correct equations from $\mathbf{M}\begin{bmatrix} x \\ x+1 \end{bmatrix} = \begin{bmatrix} X \\ mX+c \end{bmatrix}$ or equivalent. PI **Or** Obtains a correct specific point on the line $y = 2x + 8$	1.1b	A1	
Forms a correct equation in either m or c FT their two equations **Or** Multiplies **M** by a second specific point on the line $y = x + 1$	1.1a	M1	
Solves two equations in m and c Accept one incorrect equation if it has clearly come from $\mathbf{M}\begin{bmatrix} x \\ x+1 \end{bmatrix} = \begin{bmatrix} X \\ mX+c \end{bmatrix}$ or equivalent. **Or** Forms an unsimplified Cartesian equation for a straight line connecting their two points. Accept one incorrect point if it has clearly come from an attempt to find a point on the new line.	1.1a	M1	
Obtains $y = 2x + 8$	2.2a	A1	
Subtotal		**5**	

7.

3(a)	Use $F = \dfrac{1000P}{v}$ where $v = \dfrac{72000}{3600}$ $(= 20)$	M1
	Use equation of motion: $F - 900 = 0$ to give equation in P only: Eg $\frac{1000P}{20} = 900$	M1
	$P = 18$	A1
		(3)
3(b)	Use equation of motion for car **and** power equation to give equation in a only	M1
	$\dfrac{30000}{10} - 1000g\sin\alpha - 20\times10 = 1000a$ o.e.	A1
		A1
	$a = 2.4$	A1
		(4)
3(c)	$F = \dfrac{30\ 000}{U}$	M1
	Equation of motion for car	M1
	$F - 1000g\sin\alpha - 20U = 0$	A1
	Correct equation in U only oe Eg • $\frac{30000}{U} - 1000g\left(\frac{2}{49}\right) - 20U = 0$ • $U^2 + 20U - 1500 = 0$	A1
	$U = 30$	A1
		(5)