L6 Further Maths January Exam Teacher Y Mock 24-25 SOLUTIONS [53]

1.

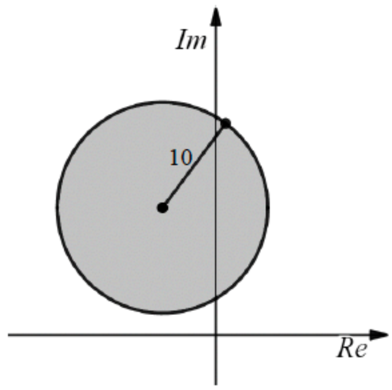
Forms the product zw	1.1a	M1	$zw=\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\mathrm{i}\right)\left(\frac{\sqrt{3}}{2}+\frac{1}{2}\mathrm{i}\right)$ $=\frac{\sqrt{6}}{4}-\frac{\sqrt{2}}{4}+\mathrm{i}\left(\frac{\sqrt{6}}{4}+\frac{\sqrt{2}}{4}\right)$ Also $zw=\cos\left(\frac{\pi}{4}+\frac{\pi}{6}\right)+\mathrm{i}\sin\left(\frac{\pi}{4}+\frac{\pi}{6}\right)$ $=\cos\frac{5\pi}{12}+\mathrm{i}\sin\frac{5\pi}{12}$ $\tan\frac{5\pi}{12}=\frac{\sin\frac{5\pi}{12}}{\cos\frac{5\pi}{12}}=\frac{\frac{\sqrt{6}}{4}+\frac{\sqrt{2}}{4}}{\frac{\sqrt{6}}{4}-\frac{\sqrt{2}}{4}}$ $=2+\sqrt{3}$
Obtains $\frac{\sqrt{6}}{4}-\frac{\sqrt{2}}{4}+\mathrm{i}\left(\frac{\sqrt{6}}{4}+\frac{\sqrt{2}}{4}\right)$	1.1b	A1	
States or uses $arg(zw)=arg(z)+arg(w)$	3.1a	M1	
States $\frac{\pi}{4}+\frac{\pi}{6}=\frac{5\pi}{12}$	1.1b	B1	
Uses their $zw=\frac{\sqrt{6}}{4}-\frac{\sqrt{2}}{4}+\mathrm{i}\left(\frac{\sqrt{6}}{4}+\frac{\sqrt{2}}{4}\right)$ to deduce an expression for $\tan\frac{5\pi}{12}$	2.2a	M1	
Completes a reasoned argument to obtain $\tan\frac{5\pi}{12}=2+\sqrt{3}$ AG	2.1	R1	
Question total		**6**	

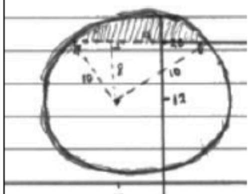
2.

1(i)	$\alpha+\beta+\gamma=\frac{3}{2},\ \alpha\beta+\alpha\gamma+\beta\gamma=\frac{5}{2}$	B1	
	$\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)=\left(\frac{3}{2}\right)^2-2\left(\frac{5}{2}\right)=\ldots$	M1	
	$=-\frac{11}{4}=-2.75$ cso	A1	
		(3)	
(ii)	$\alpha\beta\gamma=-\frac{7}{2}$ or $x=\frac{3}{w}$ used in the equation	B1	
	$\frac{3}{\alpha}+\frac{3}{\beta}+\frac{3}{\gamma}=\frac{3(\alpha\beta+\alpha\gamma+\beta\gamma)}{\alpha\beta\gamma}=\frac{3\left(\frac{5}{2}\right)}{\left(-\frac{7}{2}\right)}$ or $2\left(\frac{3}{w}\right)^3-3\left(\frac{3}{w}\right)^2+5\left(\frac{3}{w}\right)+7=0\Rightarrow 7w^3+15w^2-27w+54\{=0\}$ $\Rightarrow\ -\frac{'15'}{'7'}$	M1	
	$=-\frac{15}{7}$ cso	A1	
		(3)	
(iii)	$(5-\alpha)(5-\beta)(5-\gamma)=A\pm B(\alpha+\beta+\gamma)\pm C(\alpha\beta+\alpha\gamma+\beta\gamma)\pm(\alpha\beta\gamma)$ $=\{5^3-5^2(\alpha+\beta+\gamma)+5(\alpha\beta+\alpha\gamma+\beta\gamma)-\alpha\beta\gamma\}$ or $2(5-w)^3-3(5-w)^2+5(5-w)+7\{=0\}$ or $f(x)=A(x-\alpha)(x-\beta)(x-\gamma)\Rightarrow A=2$	M1	
	$(5-\alpha)(5-\beta)(5-\gamma)=125-25\left(\frac{3}{2}\right)+5\left(\frac{5}{2}\right)+\frac{7}{2}$ or $(5-\alpha)(5-\beta)(5-\gamma)=-\left(\frac{2\times125-3\times25+25+7}{-2}\right)$ Or $-2w^3+27w^2-125w+207\{=0\}\Rightarrow-\frac{'207'}{'-2'}$ Or $f(5)=2(5-\alpha)(5-\beta)(5-\gamma)$ $\Rightarrow(5-\alpha)(5-\beta)(5-\gamma)=\frac{f(5)}{2}$	M1	
	$=\frac{207}{2}=103.5$ cso	A1	1.1b
		(3)	
			(9 marks)

3.

DR $\alpha\beta\gamma = -(-\frac{3}{2})\ (=\frac{3}{2})$	**M1**	**1.1**	For $\alpha\beta\gamma = -(-\frac{3}{2})$.
$w = \frac{3}{2}x$	**M1**	**3.1a**	Appropriate substitution of the form $w = (\alpha\beta\gamma)x$ with **their** value of $\alpha\beta\gamma$ (need not be substituted into given cubic for this mark). Condone reciprocal e.g. $x = \frac{3}{2}w$.
$2\left(\frac{2}{3}w\right)^3 + 3\left(\frac{2}{3}w\right)^2 + 6\left(\frac{2}{3}w\right) - 3(=0)$ $16w^3 + 36w^2 + 108w - 81 = 0$	**A1**	**2.2a**	Allow any integer multiple but must have integer coefficients. Must be an equation (so must = 0 **or** terms on both sides of an equation) with all terms simplified. Condone if in terms of *x*.
	[3]		
Alternative method $\sum\alpha = -\frac{3}{2},\quad \sum\alpha\beta = 3,\quad \alpha\beta\gamma = -\left(-\frac{3}{2}\right)$	**M1**		For at least one of $\alpha\beta\gamma, \alpha+\beta+\gamma, \alpha\beta+\beta\gamma+\gamma\alpha$ correct. Can be implied by one correct coefficient in new cubic equation.
$\alpha\beta\gamma(\alpha+\beta+\gamma) = -\frac{9}{4}$ $(\alpha\beta\gamma)^2(\alpha\beta+\beta\gamma+\gamma\alpha) = \frac{27}{4}$ $(\alpha\beta\gamma)^4 = \frac{81}{16}$	**M1**		For at least two of $(\alpha\beta\gamma)^4$, $(\alpha\beta\gamma)^2(\alpha\beta+\beta\gamma+\gamma\alpha), \alpha\beta\gamma(\alpha+\beta+\gamma)$ correct – not from incorrect values of $\alpha\beta\gamma, \alpha+\beta+\gamma, \alpha\beta+\beta\gamma+\gamma\alpha$.
$w^3 + \frac{9}{4}w^2 + \frac{27}{4}w - \frac{81}{16} = 0$ $16w^3 + 36w^2 + 108w - 81 = 0$	**A1**		Allow any integer multiple but must have integer coefficients. Must be an equation (so must = 0 **or** terms on both sides of an equation) with all terms simplified. Condone if in terms of *x*.
	[3]		

5(a)(i)	$(-5, 12)$ or $-5 + 12i$	B1
(ii)	$r = 10$	B1
		(2)
(b)		B1ft
		(1)
(c)	$OC = \sqrt{5^2 + 12^2}$	M1
	$\lvert z\rvert_{\max} = \sqrt{5^2 + 12^2} + 10$	M1
	$= 23$	A1
		(3)
	Alternative $y = -\frac{12}{5}x$ and $(x+5)^2 + (y-12)^2 = 10^2$ $(x+5)^2 + \left(-\frac{12}{5}x - 12\right)^2 = 10^2 \Rightarrow x = \ldots$ Or $\tan\theta = \frac{5}{12} \Rightarrow \theta = \ldots\{22.61..\}\quad x = -5 - 10\sin\theta = \ldots$	M1
	$x = '-\frac{115}{13}' \Rightarrow y = \ldots\left\{\frac{276}{13}\right\}$ $\lvert z\rvert_{\max} = \sqrt{\left(-\frac{115}{13}\right)^2 + \left(\frac{276}{13}\right)^2}$ Or $x = '-\frac{15}{13}' \Rightarrow y = \ldots\left\{\frac{36}{13}\right\}$ $\lvert z\rvert_{\max} = \sqrt{\left(-\frac{15}{13}\right)^2 + \left(\frac{36}{13}\right)^2} + 2\times 10$	M1
	$= 23$	A1

		(3)	
(d)	$\{z : 0 \leqslant \arg(z+5-20\text{i}) \leqslant \pi\} \Rightarrow y = 20$ $\Rightarrow (x+5)^2 + 8^2 = 100 \Rightarrow x = ...$ **AND finds an angle** $\cos\theta = \dfrac{10^2+10^2-12^2}{2\times10\times10} = 0.28$ Or $a^2 = 10^2 - 8^2 \Rightarrow a = ...\{6\}\ \sin\left(\dfrac{1}{2}\theta\right) = \dfrac{6}{10}$ Or  $\cos\left(\dfrac{1}{2}\theta\right) = \dfrac{8}{10}$	M1	3.1a
	$\theta = 1.287...\text{or } 73.7^\circ$ or $\dfrac{1}{2}\theta = 0.6435...\text{or } 36.9^\circ$	A1	1.1b
	Area $= \dfrac{1}{2}\times10^2\times\theta - \dfrac{1}{2}\times12\times8$ angle in radians Area $= \pi\times10^2\times\dfrac{\theta}{360} - \dfrac{1}{2}\times12\times8$ angle in degrees or Area $= \dfrac{1}{2}\times10^2\times\theta - \dfrac{1}{2}\times10\times10\times\sin\theta$ angle in radians Area $= \pi\times10^2\times\dfrac{\theta}{360} - \dfrac{1}{2}\times10\times10\times\sin\theta$ angle in degrees Or Area $= 2\left[\dfrac{1}{2}\times10^2\times\theta - \dfrac{1}{2}\times8\times6\right]$	M1	3.1a
	= awrt 16.4	A1	1.1b
		(4)	
			(10 marks)

5.

DR $a = 0$ (is one possibility)	**B1**	1.1		
$1^4 + 1^3 + 3\times1^2 - 5\times1 = 0$ so $a = 1$ (is another possibility)	**B1**	3.1a	or eg f(1) = 0 if intent is clear but must be some justification.	
$x^4 + x^3 + 3x^2 - 5x = x(x^2(x-1) + 2x(x-1) + 5(x-1)) = x(x-1)(x^2 + 2x + 5)$ and discriminant of quadratic $= 2^2 - 4\times1\times5 = -16 < 0$ so no further real roots.	**B1**	2.3	Some justification must be given that there are no more real roots.	Allow for finding the two complex roots ($-1 \pm 2i$) - must have seen the correct $x^2 + 2x + 5$ If **B1B0B0** or **B0B0B0** then **SC1** for "$a = 1$ and no others" or "a=1, ($-1 \pm 2i$)" without justification.
$a = 0 \Rightarrow 2 + 3i$ is a root of $z^4 - 2z^3 - 10z^2 + 86z - 195$ [= 0] so $2 - 3i$ is also a root OR $a = 1 \Rightarrow 3 + 3i$ is a root of $z^4 - 4z^3 + 11z^2 + 6z + 90$ [= 0] so $3 - 3i$ is also a root	**M1**	3.1a	Condone small errors in calculation of coefficients in equation	Need both the pair of complex roots SOI and the quartic shown (allow sign slips). Only need one case for the M1.
$2 + 3i + 2 - 3i = 4$ and $(2 + 3i)(2 - 3i) = 13$ so $z^2 - 4z + 13$ is a factor OR $3 + 3i + 3 - 3i = 6$ and $(3 + 3i)(3 - 3i) = 18$ so $z^2 - 6z + 18$ is a factor	**M1**	1.1	oe eg expanding $(z - (2 + 3i))(z - (2 - 3i))$ or $(z - (3 + 3i))(z - (3 - 3i))$	Attempt to find quadratic factor from the complex roots. Only one case needed for M1 Allow with no working
$z^4 - 2z^3 - 10z^2 + 86z - 195 = (z^2 - 4z + 13)(z^2 + 2z - 15)$ OR $z^4 - 4z^3 + 11z^2 + 6z + 90 = (z^2 - 6z + 18)(z^2 + 2z + 5)$	**M1**	1.1	Attempt to factorise their quartic with their quadratic factor (at least z^3 and constant terms consistent).	Only one case needed. MUST see some evidence of factorisation here
$z^2 + 2z - 15 = 0 \Rightarrow z = -5, z = 3$ and $2 \pm 3i$ stated as roots (possibly earlier).	**A1**	2.2a	All four roots SOI for the a=0 case	**DR** so need to see evidence of where the roots came from i.e. factorisation into two quadratics If extra values of z found (from complex a or incorrect a values) then A0
$z^2 + 2z + 5 = 0 \Rightarrow z = -1 \pm 2i$ and $3 \pm 3i$ stated as roots (possibly earlier).	**A1** **[8]**	2.2a	All four roots SOI for the a=1 case	
Alternate for 1st and 2nd M mark $(a+2+3i)+(a+2-3i)=2a+4$ and $(a+2+3i)(a+2-3i)=(a+2)^2+9$ so $z^2 - (2a+4)z + [a^2+4a+13]$ is a factor	**M1**		Quadratic factor found in general case. Can also be found by expanding $[z-(a+2+3i)][z - (a+2-3i)]$	Constant term can be either $(a+2)^2+9$ or $[a^2+4a+13]$
$a = 0 \Rightarrow z^2 - 4z + 13$ is a factor OR $a = 1 \Rightarrow z^2 - 6z + 18$ is a factor	**M1**			
Alternate 2 if only the quartic in z with a=0 considered via linear factors $a = 0$ (is one possibility)	**B1**			
$1^4 + 1^3 + 3\times1^2 - 5\times1 = 0$ so $a = 1$ (is another possibility)	**B1**		or eg f(1) = 0 if intent is clear but must be some justification.	
$x^4 + x^3 + 3x^2 - 5x = x(x^2(x-1) + 2x(x-1) + 5(x-1)) = x(x-1)(x^2 + 2x + 5)$ and discriminant of quadratic $= 2^2 - 4\times1\times5 = -16 < 0$ so no further real roots.	**B1**		Some justification must be given that there are no more real roots. Allow for finding the two complex roots ($-1 \pm 2i$) (must have seen correct quadratic)	If **B1B0B0** or **B0B0B0** then **SC1** for "$a = 1$ and no others" or "a=1, ($-1 \pm 2i$)" without justification.
If a=0 quartic is $z^4 - 2z^3 - 10z^2 + 86z - 195$ f(3)=0 so $(z-3)$ is a factor $f(z) = (z-3)(z^3 + z^2 - 7z + 65)$	**M1**		Identifying linear factor and factorising	
f(-5)=0 so $(z+5)$ is a factor $f(z) = (z-3)(z+5)(z^2 - 4z + 13)$	**M1**			
So the roots are 3, -5, 2+3i, 2-3i	**A1**		Identifying 4 roots	Final 2 marks unavailable

6.

(a)	$[({}^6C_2 \times 5) + {}^6C_3] \div {}^{11}C_3$ $[= (15\times5 + 20) \div 165]$ *or* $\frac{6}{11}\times\frac{5}{10}\times\frac{5}{9}\times3+\frac{6}{11}\times\frac{5}{10}\times\frac{4}{9}$, *or* $1-\dfrac{{}^5C_3+{}^5C_2\times6}{{}^{11}C_3}$	**M1**	2.1	One term omitted from quotient (e.g. $\frac{5}{11} = 0.455$ or $\frac{7}{33} = 0.212$ or $\frac{7}{11} = 0.636$ or $\frac{29}{33} = 0.879$), or one number from product, (e.g. $\frac{3}{11} = 0.273$): M1A0
	$= \frac{19}{33}$ or 0.5757...	**A1** **[2]**	1.1	Exact or 0.576 or better Other methods: Correct answer B2, else 0
(b)	$7 \div {}^{11}C_5$ $[= 7 \div 462]$	**M1**	1.1	Or $\frac{5}{11}\times\frac{4}{10}\times\frac{3}{9}\times\frac{2}{8}\times\frac{1}{7}\times7$ (must have 6 or 7) $6 \div {}^{11}C_5$ ($= \frac{1}{77} = 0.0130$) or $7 \div {}^{11}P_5$: M1A0
	$= \frac{1}{66}$ or 0.01515...	**A1** **[2]**	2.2a	Exact or 0.015 or better, www Other methods: Correct answer B2, else 0
(c)	${}^7C_5 \div {}^{11}C_5$ $[= 21 \div 462]$	**M1**	3.1b	Either correct term seen, e.g. (5!6!)/11!
		A1	1.1	Fully correct expression
	$= \frac{1}{22}$ or 0.04545...	**A1** **[3]**	2.2a	Exact or 0.045 or better, www Other methods: Correct answer B3, else 0
(d)	$7 \times {}^6C_2 \div {}^{11}C_5$	**M1**	3.1b	${}^7P_2$, ${}^7C_2$, ${}^6C_2$, ${}^6P_2$ *or* ${}^7C_3$ in numerator, e.g. 7×6×5 or 6×5×4 (e.g. $\frac{1}{11}, \frac{1}{22}, \frac{5}{154}, \frac{5}{11}, \frac{5}{66}$) or $\frac{5}{66}$ *or* diagram showing 6 blues and 7 gaps: M1
		A1	1.1	Numerator correct
	$= \frac{5}{22}$ or 0.227272...	**A1** **[3]**	2.2a	Exact or 0.227 or better Other methods: Correct answer B3, else 0

7.

4 (a)	$X \sim \text{Geo}(\frac{1}{6})$ accept in words: geometric distribution with $p = \frac{1}{6}$	B1 **(1)**	3.3
(b)	$P(X \leqslant 3) = 1 - P(X > 3)$ or $P(X = 1) + P(X = 2) + P(X = 3)$	M1	1.1b
	$= 1-\left(\frac{5}{6}\right)^3$ or $\frac{1}{6}+\frac{5}{6}\times\frac{1}{6}+\left(\frac{5}{6}\right)^2\times\frac{1}{6}=\frac{91}{216}$ (*)	A1cso **(2)**	1.1b
(c)	$E(X) = 6$ $\text{Var}(X) = \dfrac{\frac{5}{6}}{\left(\frac{1}{6}\right)^2}$ $[=30]$	M1 A1	3.4 1.1b
	$\bar{X} \approx \sim N\left(6, \left(\sqrt{\frac{30}{64}}\right)^2\right)$	M1 A1	3.3 1.1b
	$P(5.6 < \bar{X} < 7.2) = 0.68064\ldots$	A1 **(5)**	1.1b