Cartesian equation of a line

Starter

1. Find the vector equation of the line passing through $A(1, 3, 5)$ and $B(2, 4, 6)$.

2. How do we convert $\mathbf{r} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + \lambda \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$ from vector form of the line to Cartesian form?

   Hint: Let $\mathbf{r} = \begin{pmatrix} x \\ y \end{pmatrix}$ and eliminate λ.

Notes

The vector equation of a line in 2-D which passes through the point (p_1, p_2) and is parallel to the direction vector $\begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$ is given by $\dfrac{x - p_1}{d_1} = \dfrac{y - p_2}{d_2} (= \lambda)$.

E.g. 1 Convert the vector equation $\mathbf{r} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} + \lambda \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$ to its Cartesian form.

Notice that the ***denominators*** are the components of the ***direction vectors***.

The vector equation of a line in 3-D which passes through the point (p_1, p_2, p_3) and is parallel to the direction vector $\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$ is given by $\dfrac{x - p_1}{d_1} = \dfrac{y - p_2}{d_2} = \dfrac{z - p_3}{d_3} (= \lambda)$.

What happens when one of the components of direction vector is zero?

Converting the vector equation $\mathbf{r} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} + \lambda \begin{pmatrix} d_1 \\ d_2 \\ 0 \end{pmatrix}$ to Cartesian form hits the buffers when we try to divide by zero. In essence the $z-$coordinate is always p_3 and so this appears as a separate equation.

i.e. $\mathbf{r} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} + \lambda \begin{pmatrix} d_1 \\ d_2 \\ 0 \end{pmatrix}$ is equivalent to $\dfrac{x - p_1}{d_1} = \dfrac{y - p_2}{d_2} (= \lambda)$ and $z = p_3$

E.g. 2 Convert the following to Cartesian equations:

(a) $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ -7 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$

(b) $\mathbf{r} = \begin{pmatrix} -9 \\ 5 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} -6 \\ \frac{1}{2} \\ 3 \end{pmatrix}$

(c) $\mathbf{r} = \begin{pmatrix} 7 \\ 3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ -8 \\ 5 \end{pmatrix}$

Working: (a) $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ -7 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$

$x = 2 + 4\lambda \quad \Rightarrow \quad \lambda = \dfrac{x-2}{4}$

$y = 3 - 2\lambda \quad \Rightarrow \quad \lambda = \dfrac{y-3}{-2}$

$z = -7 + \lambda \quad \Rightarrow \quad \lambda = z + 7$

The Cartesian equation is $\dfrac{x-2}{4} = \dfrac{y-3}{-2} = z + 7 (= \lambda)$

E.g. 3 Convert the following to vector equations:

(a) $\dfrac{x-2}{3} = \dfrac{y+4}{2} = \dfrac{z-1}{4}$

(b) $4(x+5) = 1 - y = \dfrac{z}{3}$

(c) $\dfrac{x+3}{-7} = 2z$ and $y = -6$

Working: (a) $\lambda = \dfrac{x-2}{3} \quad \Rightarrow \quad x = 2 + 3\lambda$

$\lambda = \dfrac{y+4}{2} \quad \Rightarrow \quad y = -4 + 2\lambda$

$\lambda = \dfrac{z-1}{4} \quad \Rightarrow \quad z = 1 + 4\lambda$

The vector equation is $\mathbf{r} = \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$

$4(x+5) = 1 - y = \dfrac{z}{3}$

Magnitude of a vector

If $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, then its magnitude is given by: $\mathbf{v} = \sqrt{v_1^2 + v_2^2}$

Unit vectors

A unit vector is a vector whose magnitude is 1.

Notation: the unit vector of $\mathbf{v}$ is $\hat{\mathbf{v}}$

$\mathbf{v} = |\mathbf{v}|\hat{\mathbf{v}}$ so $\hat{\mathbf{v}} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$

E.g. 4 Let $\mathbf{v} = 5\mathbf{i} + 12\mathbf{j}$. Find the unit vector in the direction of $\mathbf{v}$.

E.g. 5 Let $\mathbf{v} = 7\mathbf{i} - 24\mathbf{j}$. Find:
(a) the unit vector and
(b) the vector of length 75 units in the direction of $\mathbf{v}$.

Vector of specific magnitude a vector of magnitude k in the direction of $\mathbf{v}$ is $k\hat{\mathbf{v}} = k\dfrac{\mathbf{v}}{|\mathbf{v}|}$

Video A: Cartesian form of lines
Video B: Cartesian form of lines

Solutions to Starter and E.g.s

Exercise
p44 2B Qu 1i, 2i, 3i, 4i, 5-8

Summary

2-D: $\mathbf{r} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + \lambda \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$ is equivalent to $\dfrac{x - p_1}{d_1} = \dfrac{y - p_2}{d_2} (= \lambda)$

3-D: $\mathbf{r} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} + \lambda \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$ is equivalent to $\dfrac{x - p_1}{d_1} = \dfrac{y - p_2}{d_2} = \dfrac{z - p_3}{d_3} (= \lambda)$

N.B. $\mathbf{r} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} + \lambda \begin{pmatrix} d_1 \\ d_2 \\ 0 \end{pmatrix}$ is equivalent to $\dfrac{x - p_1}{d_1} = \dfrac{y - p_2}{d_2} (= \lambda)$ and $z = p_3$

A ***unit vector***, $\hat{\mathbf{v}}$, is a vector whose magnitude is 1: $\mathbf{v} = |\mathbf{v}|\hat{\mathbf{v}}$ so $\hat{\mathbf{v}} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$

Vector of specific magnitude a vector of magnitude k in the direction of $\mathbf{v}$ is $k\hat{\mathbf{v}} = k\dfrac{\mathbf{v}}{|\mathbf{v}|}$