Complex Solutions to Polynomial Equations

Starter
1. **(Review of last lesson)**
   Express the $z^3 + 3z^2 - 8z - 30$ polynomials as the products of a linear and a quadratic factor with real coefficients. Hence solve the equation when it is equal to zero.

2. **In this question you must show detailed reasoning.** (i.e. do not use your calculator)
   By solving the quadratic $z^2 - 2z + 5 = 0$ using the formula, express it as the product of two linear factors.

Notes
From the starter and from the quadratic formula, it is clear that the complex roots of a quadratic with real coefficients come in conjugate pairs.

Therefore, if $a + bi$ is root then so is $a - bi$ and vice versa.

**Roots and factors**
If $a$ is a root of a polynomial equation then $x - a$ is a factor.
In general:
\[
x = \frac{b}{a} \text{ is a root} \quad \iff \quad ax - b \text{ is a factor}
\]
\[
z = a \pm bi \text{ are roots} \quad \iff \quad x - (a + bi) \text{ and } x - (a - bi) \text{ are factors}
\]
\[
x - a - bi \text{ and } x - a + bi \text{ are factors}
\]

**E.g. 1** Find the equation of the quadratic which has one root $3 - 5i$ and whose coefficient of the squared term is 1.

**Working:**
The quadratic equation is of the form $z^2 + b z + c = 0$
If $3 - 5i$ is a root, then $3 + 5i$ is also a root.
\[
(z - (3 - 5i))(z - (3 + 5i)) = z^2 - 6z + 3^2 + 5^2
\]
\[
= z^2 - 6z + 34
\]
So the equation is $z^2 - 6z + 34 = 0$

**N.B.** Remember $(a + bi)(a - bi) = a^2 + b^2$

**E.g. 2** The equation $z^3 - 7z^2 + 16z - 10 = 0$ has a root $3 + i$. Find the other two roots of the equation.

Questions can also require the use of the factor theorem, similar to question 1 from the starter.

**E.g. 3** **In this question you must show detailed reasoning.**
Solve the equation $z^3 - 3z^2 + z + 5 = 0$.

Video: Solving cubic equations
Video: Solving quartic equations

Solutions to Starter and E.g.s
Exercise
p147 5B Qu 1i, 2i, 3i, 4-9

Summary
Complex roots appear as conjugate pairs: if $a + bi$ is root then so is $a - bi$

$x = \frac{b}{a}$ is a root $\iff$ $ax - b$ is a factor

$z = a \pm bi$ are roots $\iff$ $x - (a + bi)$ and $x - (a - bi)$ are factors