# Determinant and Inverse of 3 by 3 Matrices

**Starter**

1. Find the matrix $\mathbf{X}$ such that $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \mathbf{X}^{-1} = \begin{pmatrix} -2 & 9 \\ -2 & 14 \end{pmatrix}$.

2. If $\mathbf{A}$ is the matrix $\begin{pmatrix} 7 & 4 \\ 6 & 3 \end{pmatrix}$, show that $\mathbf{A}^2 - 10\mathbf{A} - 3\mathbf{I} = 0$. Hence find $\mathbf{A}^{-1}$.

   ***Hint:*** Remember $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$

**Notes**

***Determinant of a 3 by 3 matrix***

The determinant of a $3$ by $3$ matrix uses the determinant of a $2$ by $2$ matrix three times.

Let $\mathbf{M} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$

Here is the formula to find the determinant of a $3$ by $3$ matrix:

$$\det \mathbf{M} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

To understand it, we consider each part separately:

$$\det \mathbf{M} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$\det \mathbf{M} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$\det \mathbf{M} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

This matrix shows the signs that go with each $2$ by $2$ determinant: $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

Alternative formula: $\det \mathbf{M} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$

***E.g. 1*** Find the determinant of the matrix $\begin{pmatrix} 1 & -2 & 3 \\ 1 & -1 & 2 \\ -2 & 4 & -5 \end{pmatrix}$.

***Key facts to help you calculate the determinant of a $3$ by $3$ matrix***

Here are a few shortcuts to help speed up the calculation of the determinant. The aim is to simplify the calculation, usually by getting as many zeros in the top row as possible.

- When two rows or columns are swapped over, the determinant changes sign.

  ***E.g.*** $\begin{vmatrix} 1 & 6 & 7 \\ 3 & 0 & 0 \\ -8 & 2 & 3 \end{vmatrix} = -\begin{vmatrix} 3 & 0 & 0 \\ 1 & 6 & 7 \\ -8 & 2 & 3 \end{vmatrix} = 3\begin{vmatrix} 6 & 7 \\ 2 & 3 \end{vmatrix} = \ldots$

- If a row or column has a common factor, it is a factor of the determinant.

  ***E.g.*** $\begin{vmatrix} 7 & 2 & -5 \\ 4 & 12 & 6 \\ 1 & -8 & 4 \end{vmatrix} = 2\begin{vmatrix} 7 & 2 & -5 \\ 2 & 6 & 3 \\ 1 & -8 & 4 \end{vmatrix}$

- To any row (or column) can be added multiples of any other row (or column) without altering the value of the determinant

  ***E.g.*** Consider $\begin{vmatrix} 5 & 2 & 2 \\ -2 & 1 & 1 \\ 8 & -6 & 3 \end{vmatrix}$.

  $R_1 - 2R_2$ gives: $\begin{vmatrix} 5 & 2 & 2 \\ -2 & 1 & 1 \\ 8 & -6 & 3 \end{vmatrix} = \begin{vmatrix} 9 & 0 & 0 \\ -2 & 1 & 1 \\ 8 & -6 & 3 \end{vmatrix} = 9\begin{vmatrix} 1 & 1 \\ -6 & 3 \end{vmatrix} = \ldots$

- If a matrix has two identical rows or columns then the determinant is zero.

  ***E.g.*** $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 4 & -9 & 7 \end{vmatrix} = 0$

- The determinant of a matrix is equal to the determinant of the transposed matrix i.e. det $\mathbf{M} =$ det $\mathbf{M}^T$. Therefore, if there is a zero in the first column, transpose and calculate the determinant or use the alternative formula.

  ***E.g.*** $\begin{vmatrix} 2 & 4 & 1 \\ 0 & -5 & 3 \\ 0 & 7 & 6 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 \\ 4 & -5 & 7 \\ 1 & 3 & 6 \end{vmatrix} = 2\begin{vmatrix} -5 & 7 \\ 3 & 6 \end{vmatrix} = \ldots$

***E.g. 2*** Using the shortcuts above, find the determinants of these matrices:

(a) $\begin{pmatrix} 1 & -2 & 3 \\ 1 & -1 & 2 \\ -2 & 4 & -5 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ -2 & -2 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & -1 & 3 \\ 1 & 2 & 1 \\ 3 & -4 & 5 \end{pmatrix}$

***Working:*** (a) Transpose: $\begin{vmatrix} 1 & -2 & 3 \\ 1 & -1 & 2 \\ -2 & 4 & -5 \end{vmatrix} = \begin{vmatrix} 1 & 1 & -2 \\ -2 & -1 & 4 \\ 3 & 2 & -5 \end{vmatrix}$

$R_1 + R_2$: $\begin{vmatrix} -1 & 0 & 2 \\ -2 & -1 & 4 \\ 3 & 2 & -5 \end{vmatrix}$

$$\begin{vmatrix} -1 & 0 & 2 \\ -2 & -1 & 4 \\ 3 & 2 & -5 \end{vmatrix} = -1\begin{vmatrix} -1 & 4 \\ 2 & -5 \end{vmatrix} + 2\begin{vmatrix} -2 & -1 \\ 3 & 2 \end{vmatrix}$$
$$= 3 + 2 \times (-1) = 1$$

***Inverse of a 3 by 3 process***
***Non-calculator***
***Matrix of cofactors***

The first step to find the inverse of a $3$ by $3$ matrix by hand is to calculate the matrix of ***cofactors***. The cofactor, $A_1$, of $a_1$ is the $2$ by $2$ determinant left after the the row and column in which $a_1$ lies have been removed.

***N.B.*** Remember to use the matrix of signs: $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

Here are the complete cofactors:

$A_1 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ $B_1 = -\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$ $C_1 = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

$A_2 = -\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}$ $B_2 = \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$ $C_2 = -\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$

$A_3 = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$ $B_3 = -\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$ $C_3 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

***Success Criteria — finding the inverse of a 3 by 3 matrix (non-calculator)***

Let $M = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$

1. Find the determinant of the matrix, det $\mathbf{M}$.
2. Find the matrix of cofactors $\mathbf{C} = \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix}$
3. Transpose the matrix of cofactors to get $\mathbf{C}^T = \begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}$

4. Divide the matrix of cofactors by the determinant:

$$\mathbf{M}^{-1} = \frac{1}{\det \mathbf{M}} C^T = \frac{1}{\det \mathbf{M}} \begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}$$

From this we can see that $\mathbf{M} \times \mathbf{C}^T = \begin{pmatrix} \det \mathbf{M} & 0 & 0 \\ 0 & \det \mathbf{M} & 0 \\ 0 & 0 & \det \mathbf{M} \end{pmatrix}$

***N.B.*** If the matrix is singular i.e. $\det \mathbf{M} = 0$, then the inverse of the matrix does not exist.

***E.g. 3*** Find the inverse of the matrix $\mathbf{M} = \begin{pmatrix} 4 & 1 & -1 \\ 3 & -1 & 2 \\ 4 & 0 & -3 \end{pmatrix}$

***Working:*** $\det \mathbf{M} = 25$

Matrix of cofactors is $\mathbf{C} = \begin{pmatrix} 3 & 17 & 4 \\ 3 & -8 & 4 \\ 1 & -11 & -7 \end{pmatrix}$ so $\mathbf{C}^T = \begin{pmatrix} 3 & 3 & 1 \\ 17 & -8 & -11 \\ 4 & 4 & -7 \end{pmatrix}$

$$\mathbf{M}^{-1} = \frac{1}{25} \begin{pmatrix} 3 & 3 & 1 \\ 17 & -8 & -11 \\ 4 & 4 & -7 \end{pmatrix}$$

***E.g. 4*** Find the inverse of the matrix $\begin{pmatrix} 1 & -2 & 3 \\ 1 & -1 & 2 \\ -2 & 4 & -5 \end{pmatrix}$.

***Hint:*** You have already found its determinant in ***E.g. 2***

***Using a calculator to find the inverse of a $3$ by $3$ matrix***

Menu >> 4 : Matrix >> Press 1–4 to select Matrix A—D >> Choose the number of rows >> Choose the number of columns >> Enter the elements of the matrix >> OPTN >> 3: Matrix Calc >> OPTN

***Finding the determinant*** >> (Down arrow) >> 2 : Determinant >> OPTN >> (Choose the matrix) >> Press ) >> Press =

***Finding the inverse*** >> Press 3–6 to select Mat A—D >> Press $x^{-1}$

**Video:** Using a calculator to find the determinant and/or the inverse of matrices

***E.g. 5*** Use your calculator to find the inverse of these matrices:

(a) $\begin{pmatrix} 1 & -2 & -3 \\ 2 & -1 & -4 \\ 3 & -3 & -5 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & -2 & -1 \\ 2 & 1 & 5 \\ 3 & -2 & 3 \end{pmatrix}$

***Working:*** (a) $\begin{pmatrix} -\frac{7}{6} & -\frac{1}{6} & \frac{5}{6} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$

***E.g. 6*** Using your calculator, solve for $\mathbf{X}$ the equation $\begin{pmatrix} 1 & 3 & 2 \\ 4 & 0 & -1 \\ 2 & 3 & -3 \end{pmatrix}\mathbf{X} = \begin{pmatrix} 0 & -6 & 3 \\ 21 & 6 & -9 \\ -9 & 5 & -4 \end{pmatrix}$.

Video: Determinant of 3 by 3 matrices
Video A: Inverse of 3 by 3 matrices
Video B: Inverse of 3 by 3 matrices
Video: Using a calculator to find the determinant and/or the inverse of matrices

Solutions to Starter and E.g.s

**Exercise**
p27 1E Qu 1i, 2i, 3i, 4i, 5-9

**Summary**

$$\det \mathbf{M} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

A singular matrix is when the determinant equals zero i.e. det $\mathbf{M} = 0$

- When two rows or columns are swapped over, the determinant changes sign.
- If a row or column has a common factor, it is a factor of the determinant.
- To any row (or column) can be added multiples of any other row (or column) without altering the value of the determinant
- If a matrix has two identical rows or columns then the determinant is zero.
- The determinant of a matrix is equal to the determinant of the transposed matrix i.e. det $\mathbf{M} =$ det $\mathbf{M}^T$. Therefore, if there is a zero in the first column, transpose and calculate the determinant or use the alternative formula.

Cofactors:

$$A_1 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} \qquad B_1 = -\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} \qquad C_1 = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$A_2 = -\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} \qquad B_2 = \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} \qquad C_2 = -\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$$

$$A_3 = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \qquad B_3 = -\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \qquad C_3 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

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