Discrete Uniform Distributions

Starter

1. **(Review of last lesson)**

   The discrete random X has p.d.f. give by $P(X=x)=\dfrac{3x+1}{22}$ for $x=0,\ 1,\ 2,\ 3$. Find:

   (a) $E(X)$ and $E(X^2)$
   (b) Var(X)
   (c) $E(3X-2)$
   (d) $E(2X^2+4X-3)$
   (e) Var$(4X)$
   (f) Var$(5X+7)$

2. Without writing out all the terms, calculate how many terms are in the following linear sequences?

   (a) $5,\ 7,\ 9,...,\ 37$
   (b) $3,\ 7,\ 11,...,\ 31$
   (c) $a,\ a+d,\ a+2d,\ ...,\ b$

Notes

As the name suggests, a discrete uniform distribution can take a countable number of values and the probability of each value is the same.
For example, the probability distribution of a dice roll is a discrete uniform distribution.

Notation

$X\sim$ U(n) means that the random variable X is uniform and $P(X=x)=\dfrac{1}{n}$ for the n values of x.

N.B. If the values of x are not stated, assume $x=1,\ 2,\ 3,\ ...,\ n$

For example, $X\sim$ U(4) for $x=1,\ 2,\ 3,\ 4$ means

x:	1	2	3	4
$P(X=x)$:	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

E.g. 1 Let X be a discrete uniform distribution which can take the values $6,\ 9,\ 12,\ \ldots,\ 60$. Find $P(X=48)$.

E.g. 2 The random variable X can take the values $a, a+d, a+2d, \ldots, b$. Find an expression for $P(X=x)$ in terms of a, b and d.

Working: Number of terms $=\dfrac{b-a}{d}+1=\dfrac{b-a}{d}+\dfrac{d}{d}=\dfrac{b-a+d}{d}$

$\therefore P(X=x)=\dfrac{d}{b-a+d}$

In general, if X can take the values $a, a+d, a+2d, \ldots, b$ then $P(X=x)=\dfrac{d}{b-a+d}$

Mean and variance

E.g. 3 Given that the sum of the first n integers is $\frac{1}{2}n(n+1)$ and that the sum of the first n square numbers is $\frac{1}{6}n(n+1)(2n+1)$, find the mean and variance of $X \sim \text{U}(n)$.
Hint: remember if the values of x are not stated, assume $x = 1,2,3,\ldots,n$

Working:

$x:$	1	2	3	…	n
$P(X=x):$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	…	$\frac{1}{n}$

$$\begin{aligned}\text{Mean } &= (1+2+3+\ldots+n)\times\frac{1}{n}\\ &= \frac{1}{2}n(n+1)\times\frac{1}{n}\\ &= \frac{1}{2}(n+1)\end{aligned}$$

$$\begin{aligned}\text{Variance } &= E(X^2)-E^2(X)\\ &= (1^2+2^2+3^2+\ldots+n^2)\times\frac{1}{n}-\left(\frac{n+1}{2}\right)^2\\ &= \frac{1}{6}n(n+1)(2n+1)\times\frac{1}{n}-\left(\frac{n+1}{2}\right)^2\\ &= \frac{1}{6}(n+1)(2n+1)-\left(\frac{n+1}{2}\right)^2\\ &= \frac{1}{12}(n+1)\left[2(2n+1)-3(n+1)\right]\\ &= \frac{1}{12}(n+1)(n-1)\\ &= \frac{1}{12}(n^2-1)\end{aligned}$$

Summary

- $X \sim \text{U}(n)$ means $P(X=x)=\frac{1}{n}$ for the n values of x.
- Mean $=\frac{1}{2}(n+1)$
- Variance $=\frac{1}{12}(n^2-1)$

E.g. 4 Find the mean and variance for $X \sim \text{U}(31)$

E.g. 5 An icosahedron is rolled (i.e. a $20-$sided dice). Find the mean and standard deviation.
N.B. Icosahedron is pronounced "i-coss-a-hee-dron".

Video: Discrete uniform distributions

Solutions to Starter and E.g.s

Exercise
p29 2C Qu 1i, 2-8, (9-10 red)

Summary

$X \sim \mathrm{U}(n)$ means $P(X = x) = \dfrac{1}{n}$ for the n values of x.

Mean $= \dfrac{1}{2}(n+1)$

Variance $= \dfrac{1}{12}(n^2 - 1)$

Linear sequence

$a, a+d, a+2d, \ldots, b$

Number of terms $= \dfrac{b-a+d}{d}$

$\therefore P(X = x) = \dfrac{d}{b-a+d}$