Division and Complex Conjugates

Starter

1. **(Review of last lesson)** Given that $(1+5\text{i})A-2B=3+7\text{i}$, find A and B if:
   (a) A and B are both real
   (b) A and B are both complex.

2. **(Review of last lesson)** Find the square root of $-7+24\text{i}$.

3. If $p=2+3\text{i}$ and $q=2-3\text{i}$, express the following in the form $a+b\text{i}$, where a and b are real numbers.
   (a) $p+q$ (b) $p-q$ (c) pq (d) $(p+q)(p-q)$
   (e) p^2-q^2 (f) p^2+q^2 (g) $(p+q)^2$ (h) $(p-q)^2$

Notes

Complex numbers $p=2+3\text{i}$ and $q=2-3\text{i}$ are called complex conjugates because they are of the form $a+b\text{i}$ and $a-b\text{i}$.

Notation

The complex conjugate of the complex number z is denoted z^*.

If $z=x+y\text{i}$ then $z^*=x-y\text{i}$

So from question 2 of the starter, $p=q^*$ and $q=p^*$.

E.g. 1 Let $z=x+y\text{i}$ where x and y are real numbers. Find:
(a) z^* (b) $z+z^*$ (c) $z-z^*$ (d) zz^*

N.B. The ***product of a complex number and its conjugate is a real number*** i.e. zz^* is real — this is an important result that will be used later in the lesson.

E.g. 2 Write down z^* given that $z=$:
(a) $2+4\text{i}$ (b) $3-6\text{i}$ (c) $-5+2\text{i}$
(d) $2\text{i}-4$ (e) 6 (f) $-3\text{i}+7$

Complex conjugates and the solutions to quadratics

If a quadratics equation with real coefficients has complex roots, then these roots come in ***conjugate pairs***.

For example, if $x+y\text{i}$ is a root of the equation $az^2+bz+c=0$, where a, b and c are real, then $x-y\text{i}$ is also a root i.e. the roots are of the form $x\pm y\text{i}$.

Dividing by a complex number

Dividing by a complex number employs a similar method to rationalising a denominator.

E.g. 3 (a) Rationalise the denominator of $\dfrac{5+\sqrt{3}}{4+\sqrt{3}}$.

(b) Using a similar method to (a), express $\dfrac{5+3\text{i}}{4+3\text{i}}$ in the form $\dfrac{a+b\text{i}}{c}$ where a, b and c are real numbers.

Working: (a)

$$\frac{5+\sqrt{3}}{4+\sqrt{3}} = \frac{5+\sqrt{3}}{4+\sqrt{3}} \times \frac{4-\sqrt{3}}{4-\sqrt{3}}$$
$$= \frac{20-5\sqrt{3}+4\sqrt{3}-3}{16-3}$$
$$= \frac{17-\sqrt{3}}{13}$$

(b)

$$\frac{5+3\text{i}}{4+3\text{i}} = \frac{5+3\text{i}}{4+3\text{i}} \times \frac{4-3\text{i}}{4-3\text{i}}$$
$$= \frac{20-15\text{i}+15\text{i}+9}{16+9}$$
$$= \frac{29-3\text{i}}{25}$$

We have seen that $z \times z^*$ gives a real number so ***by multiplying the denominator by its complex conjugate the denominator becomes a real number***.

In general if $z_2 = x + y\text{i}$ $\quad \dfrac{z_1}{z_2} = \dfrac{z_1}{z_2} \times \dfrac{z_2^*}{z_2^*} = \dfrac{z_1 \times z_2}{x^2+y^2}$

E.g. 4 Express $\dfrac{2-7\text{i}}{1+2\text{i}}$ in the form $a + b\text{i}$ where a and b are real.

E.g. 5 Given that a and b are real and $(a+b\text{i})(3+4\text{i}) = 3-4\text{i}$ find a and b.

Video: Complex conjugates
Video: Division of complex numbers

Solutions to Starter and E.g.s

Exercise
p117 4B Qu 1i, 2i, 3-6, 7i, 8-14

Summary
The complex conjugates of the z is denoted z^*, where if $z = x + y\text{i}$ then $z^* = x - y\text{i}$.
If $z = x + y\text{i}$ is a root of an equation then so is $z^* = x - y\text{i}$.
$z \times z^* = |z|^2$