Geometric Distributions

Starter

1. (Review of last lesson) If the random variable X is such that $X \sim \text{B}(10, p)$, where $p < \frac{1}{2}$ and $\text{Var}(X) = \frac{15}{8}$, find:
   (a) the value of p,
   (b) $E(X)$ and
   (b) $P(X = 2)$.

2. A sports magazine is giving away signed photos of famous swimmers. It randomly places 15 photos in every 100 magazines. Let X be the number of magazines you purchase until you get a signed photo (i.e. the first $X - 1$ magazines do not have signed photos in).
   (a) Find the probability of getting a signed photo in the 3rd magazine you purchase but not in the first 2 i.e. find $P(X = 3)$.
   (b) Explain what $P(X = 20)$ means in this context.
   (c) Calculate $P(X = 10)$.
   (d) Write down an expression for $P(X = x)$.
   (e) Calculate the probability that you need to buy 1, 2, 3 or 4 magazines to get a signed photo.

Notes

Question 2 from the starter is an example of a geometric distribution. It is similar to the binomial distribution in that:

- There are only two outcomes: “success” or “failure”
- The outcome of one trial does not depend on the others (i.e. trials are independent)
- The probability of success is constant for each trial

The difference is the number of trials. With the binomial distribution, there are a fixed number of trials but for the geometric distribution the number of trials stops after the first success
The geometric distribution has only one parameter — the probability of success, p.

Notation

$X \sim \text{Geo}(p)$, where p is the probability of success
$P(X = x)$ means there are $x - 1$ fails and then one is success at the end.

Refer to ‘AS FM Distributions’ spreadsheet.

E.g. 1 A child has a spinner with five equal sections, numbered 1 to 5. To start a game, they must spin a 4 or a 5. The child spins until a 4 or 5 is spun. Calculate the probability that they need:

(a) exactly 4 spins
(b) at most 2 spins
(c) at least 7 spins
(d) at most 12 spins
(e) How many spins are required to be $99\,\%$ sure of getting a 4 or 5?

Working: (a)

(b)

(c) At least 7 spins is $P(X \geq 7) = P(X = 7) + P(X = 8) + \ldots$
However, it is easiest to realise that $P(X \geq 7)$ is equivalent to having had 6 fails.

$$\therefore P(X \geq 7) = 0.6^6 = 0.46656 = \frac{729}{15625}$$

(d) $P(X \leq 12) = P(X = 1) + P(X = 2) + \ldots + P(X = 12)$ but it is easier to consider the complementary event.
i.e. $P(X \leq 12) = 1 - P(X \geq 13)$
Using similar thinking to (c), $P(X \geq 13)$ is equivalent to 12 fails.
$\therefore P(X \leq 12) = 1 - 0.6^{12} = 0.998$ (3 s.f.)

(e) We need to find x such that $P(X \leq x) > 0.99$.

Using similar thinking to (d): $1 - P(X \geq x) > 0.99$

$$1 - 0.6^x > 0.99$$
$$0.6^x < 0.01$$

Solve using logs: $\ln 0.6^x < \ln 0.01$

$$x \ln 0.6 < \ln 0.01$$

$\ln 0.6 < 0$ ***so inequality changes direction:*** $x > \dfrac{\ln 0.01}{\ln 0.6}$

$$x > 9.015$$

So 10 spins are required.

E.g. 2 Let $X \sim \text{Geo}(p)$.

(a) Copy and complete the table.

x:	1	2	3	4	x
$P(X=x)$:					

(b) Find $P(X \geq x)$

(c) Find $P(X \leq x)$

Working: (a)

x:	1	2	3	4	x
$P(X=x)$:	p	$(1-p)p$	$(1-p)^2 p$	$(1-p)^3 p$	$(1-p)^{x-1} p$

(b) $P(X \geq x) = P(X = x) + P(X = x+1) + P(X = x+2) + \dots$
But it is easier to think of $P(X \geq x)$ as there having been $x - 1$ fails.
$\therefore P(X \geq x) = (1-p)^{x-1}$

(c)

In general:

- $P(X = x) = (1-p)^{x-1} p$
- $P(X \geq x) = (1-p)^{x-1}$ $\qquad P(X \geq x) \equiv (x-1)$ fails
- $P(X \leq x) = 1 - P(X \geq x+1)$
  $= 1 - P(x \text{ fails})$
  $= 1 - (1-p)^x$
- $E(X) = \dfrac{1}{p}$
- $\text{Var}(X) = \dfrac{1-p}{p^2}$

E.g. 3 It is given that the random variable, T, which can take values $1, 2, 3, \dots$ has a geometric distribution and $P(T = 1) = 0.15$. Calculate:
(a) $P(T \geq 8)$ (b) $P(T > 11)$ (c) $P(T \leq 9)$ (d) $P(T < 15)$.
Give your answers to 4 s.f.

Video: Geometric distributions
Video: Least trials before success

Solutions to Starter and E.g.s

Exercise
p33 2E Qu 1i, 2i, 3-9, (10 red)

Summary
$X \sim \text{Geo}(p)$, where p is the probability of success
$P(X = x)$ means there are $x - 1$ fails and then one is success at the end.
$P(X = x) = (1-p)^{x-1} p$
$P(X \geq x) = (1-p)^{x-1}$ $\qquad P(X \geq x) \equiv (x-1)$ fails

$$\begin{aligned} P(X \le x) = 1 - P(X \ge x + 1) \\ = 1 - P\big(x \text{ fails}\big) \\ = 1 - (1 - p)^x \end{aligned}$$