Induction and divisibility

Starter

1. Prove that $\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ 3^n - 1 & 3^n \end{pmatrix}$ for any positive integer using induction.

Notes

The divisibility of an algebraic expression can also be proved using proof by induction.

There are two ways to do the manipulative steps when proving divisibility.

The first method uses the fact that if an expression is divisible by an integer d, you can write it as $M \times d$ for some integer M.

The second method is more difficult and requires more complicated algebra. Both are shown in the examples below.

E.g. 1 Prove that, for positive integers n, $8^n - 3^n$ is always divisible by 5.

Working:

(Proposition)
Let $P(n)$ be the proposition that $8^n - 3^n$ is always divisible by 5.
(Prove the basic case)
When $n = 1$, $8^1 - 3^1 = 5$ which is divisible by 5
Therefore $P(1)$ is true.
(Inductive step — n replaced by k)
Assume that $P(k)$ is true i.e. $8^k - 3^k$ is always divisible by 5
So $8^k - 3^k = 5M$ where M is a positive integer
(Inductive step — consider the next term)
$P(k+1)$ is the term $8^{k+1} - 3^{k+1}$.
(Inductive step — manipulation to show this is also divisible by 5)
$P(k+1) = 8 \times 8^k - 3 \times 3^k$.
But $8^k = 3^k + 5M$
$P(k+1) = 8 \times (3^k + 5M) - 3 \times 3^k = 5 \times 3^k + 40M = 5(3^k + 8M)$
which is divisible by 5.
(Completion)
But this is $P(k)$ with k replaced by $k+1$.
Therefore, if $P(k)$ is true, then $P(k+1)$ is also true.
$P(1)$ is true and if $P(k)$ is true, then $P(k+1)$ is true. Using the principle of mathematical induction, $P(n)$ is true for all positive integers.

Alternative inductive manipulation steps
(Inductive step — manipulation to show this is also divisible by 5)

$$\begin{aligned} P(k+1) &= 8 \times 8^k - 3 \times 3^k \\ &= (5+3)8^k - 3 \times 3^k \\ &= 5 \times 8^k + 3 \times 8^k - 3 \times 3^k \\ &= 5 \times 8^k + 3(8^k - 3^k) \\ &= 5 \times 8^k + 3P(k) \end{aligned}$$

Since we assume $P(k)$ is true, both terms are divisible by 5.
Hence $P(k+1)$ is divisible by 5

E.g. 2 Prove by mathematical induction that $6^n + 4$ is divisible by 5 for positive integers values of n.

E.g. 3 Prove that if $n \geq 1$ is a positive integer, then $13^n - 6^n$ is divisible by 7.

Video: Proof by induction (divisibility)

Solutions to Starter and E.g.s

Exercise
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Summary
There are two ways to do the manipulative steps when proving divisibility.
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The second method is more difficult and requires more complicated algebra.