Induction and matrices

Starter

1. Given that $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, $\mathbf{k} \times \mathbf{r} = \mathbf{p}$ and $\mathbf{r} \times \mathbf{p} = \mathbf{k}$, where a, b and c are constants, find an expression connecting a and b and the value of c.

Notes

Proof by induction is a formal way to prove an algebraic result.

You are always given the algebraic statement that must be proved and there are four stages to the proof.

1. ***(Proposition)*** Statement of the proposition $P(n)$
2. ***(Prove the basic case)*** Prove the result for $n = 1$.
3. ***(Inductive steps)*** Assume the result is true for k and then prove that if the result for k is true then the result is also true for $k + 1$
4. ***(Completion of the proof)*** "But this is $P(k)$ with k replaced by $k + 1$. Therefore, if $P(k)$ is true, then $P(k + 1)$ is also true. $P(1)$ is true and if $P(k)$ is true, then $P(k + 1)$ is true. Using the principle of mathematical induction, $P(n)$ is true for all positive integers."

The key step is the inductive step. The final line of the algebraic manipulation must be the same formula as $P(k)$ but with k replaced by $k + 1$. It is a good idea to write down in pencil in the margin what it is that you are aiming for.

E.g. 1 Let $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$. Prove that $\mathbf{A}^n = \begin{pmatrix} 1 & 0 \\ 5n & 1 \end{pmatrix}$ for all natural numbers n.

Working:

(Proposition)

Let $P(n)$ be the proposition that if $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$ then $\mathbf{A}^n = \begin{pmatrix} 1 & 0 \\ 5n & 1 \end{pmatrix}$

(Prove the basic case)

When $n = 1$, $\mathbf{A}^1 = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 5 \times 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$

Therefore $P(1)$ is true.

(Inductive step — n replaced by k)

Assume that $P(k)$ is true i.e. if $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$ then $\mathbf{A}^k = \begin{pmatrix} 1 & 0 \\ 5k & 1 \end{pmatrix}$

(Inductive step — multiply both sides by $\mathbf{A}$ to get $\mathbf{A}^{k+1}$)

$$\mathbf{A}^{k+1} = \begin{pmatrix} 1 & 0 \\ 5k & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$$

(Algebraic manipulation until RHS is k replace by $k + 1$)

Multiplying the matrices: $\mathbf{A}^{k+1} = \begin{pmatrix} 1 & 0 \\ 5k+5 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 5(k+1) & 1 \end{pmatrix}$

(Completion)

But this is $P(k)$ with k replaced by $k + 1$.
Therefore, if $P(k)$ is true, then $P(k + 1)$ is also true.
$P(1)$ is true and if $P(k)$ is true, then $P(k + 1)$ is true. Using the principle of mathematical induction, $P(n)$ is true for all positive integers.

E.g. 2 Let $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Calculate the matrices $\mathbf{A}^2$ and $\mathbf{A}^3$. Make a conjecture about the matrix $\mathbf{A}^n$ and prove it by induction.

E.g. 3 Prove that $\begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}^n = \begin{pmatrix} 1-3n & 9n \\ -n & 1+3n \end{pmatrix}$ using induction.

Video: Proof by induction (matrices)

Solutions to Starter and E.g.s

Exercise
p170 6A Qu 1-8

Summary
There are four stages to proof by induction:

1. ***(Proposition)*** Statement of the proposition $P(n)$
2. ***(Prove the basic case)*** Prove the result for $n = 1$.
3. ***(Inductive steps)*** Assume the result is true for k and then prove that if the result for k is true then the result is also true for $k + 1$
4. ***(Completion of the proof)*** "But this is $P(k)$ with k replaced by $k + 1$. Therefore, if $P(k)$ is true, then $P(k + 1)$ is also true. $P(1)$ is true and if $P(k)$ is true, then $P(k + 1)$ is true. Using the principle of mathematical induction, $P(n)$ is true for all positive integers."

The key step is the inductive step. The final line of the algebraic manipulation must be the same formula as $P(k)$ but with k replaced by $k + 1$. It is a good idea to write down in pencil in the margin what it is that you are aiming for.