Intersection of two lines

Starter

1. Convert $\mathbf{r} = (2 - 3\lambda)\mathbf{i} + (4 + 2\lambda)\mathbf{j} + \lambda\mathbf{k}$ to Cartesian form:

2. Write down the Cartesian equation of the line that passes through the point $(5, -4, 0)$ and is parallel to the vector $\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$.

3. Find the coordinates of the point of intersection between the lines with equations $\mathbf{r} = \begin{pmatrix} -3 \\ -1 \\ 7 \end{pmatrix} + s\begin{pmatrix} -4 \\ 2 \\ 5 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} -13 \\ -7 \\ 8 \end{pmatrix} + t\begin{pmatrix} 7 \\ 2 \\ -3 \end{pmatrix}$.

Notes

Success criteria — finding the point of intersection

1. Put the two equations equal to each other.
2. Form one equation for each of the components, $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.
3. Solve two of the equations simultaneously to find a value each for the two scalars.
4. Substitute the values into the 3rd equation to check that it is consistent.
   Equation consistent ⇒ lines intersect
   Equation not consistent ⇒ lines are parallel or skew (in 3-D)
5. Substitute one of the unknowns into the correct equation.

N.B. In 2-D, if lines do not intersect, they are parallel.
In 3-D, if lines do not intersect, they are either parallel or skew.

E.g. 1 Find the coordinates of the point of intersection between the lines with equations

(a) $\mathbf{r} = \begin{pmatrix} 5 \\ 4 \\ 18 \end{pmatrix} + s\begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} -12 \\ 3 \\ 19 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix}$

(b) $\mathbf{r} = \begin{pmatrix} 4 \\ -4 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} -19 \\ -22 \\ 1 \end{pmatrix} + t\begin{pmatrix} 5 \\ 6 \\ -1 \end{pmatrix}$

Working: (a) Equate the lines: $\begin{pmatrix} 5 \\ 4 \\ 18 \end{pmatrix} + s\begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} -12 \\ 3 \\ 19 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix}$

Equating components:

$\mathbf{i}$: $5 + 3s = -12 + 2t \quad \Rightarrow \quad 3s - 2t = -17$

$\mathbf{j}$: $4 - s = 3 + t \quad \Rightarrow \quad s + t = 1$

$\mathbf{k}$: $18 + 5s = 19 - 4t \quad \Rightarrow \quad 5s + 4t = 1$

Solving the $\mathbf{i}$ and $\mathbf{j}$ equations simultaneously: $s = -3, t = 4$

Check values work in the $\mathbf{k}$ equation: $5 \times (-3) + 4 \times 4 = 1$ ✓

Substitute either the $s-$ or the $t-$value in the relevant equation:

Either… $s = -3$ $\mathbf{r} = \begin{pmatrix} 5 \\ 4 \\ 18 \end{pmatrix} - 3\begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} -4 \\ 7 \\ 3 \end{pmatrix}$

…or… $t = 4$: $\mathbf{r} = \begin{pmatrix} -12 \\ 3 \\ 19 \end{pmatrix} + 4\begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} = \begin{pmatrix} -4 \\ 7 \\ 3 \end{pmatrix}$

The point of intersection between the lines is $(-4, 7, 3)$.

Video: Intersecting and skew lines

Parallel, intersecting and skew lines EQ

Solutions to Starter and E.g.s

Exercise

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Summary

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3. Solve two of the equations simultaneously to find a value each for the two scalars.
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