Introduction to Complex Numbers (triple)

Starter

1. (Review of GCSE material)
   Solve the equation $x^2 - 2x + 5 = 0$ using the quadratic formula.

Notes

Armed with our current knowledge we cannot solve the equation $x^2 - 2x + 5 = 0$, even if we use the quadratic formula, because we cannot square root −16. In order to allow us to solve such equations mathematicians invented ***complex*** or ***imaginary numbers***.

The square root of −1 is given the letter i:

$$\text{i} = \sqrt{-1}$$

So for the question from the starter, $x = \dfrac{2 \pm \sqrt{16 \times -1}}{2}$

$$x = \frac{2 \pm 4\sqrt{-1}}{2}$$

$$x = \frac{2 \pm 4\text{i}}{2}$$

$$x = 1 \pm 2\text{i}$$

Surprisingly imaginary numbers have an amazing list of applications including electrical engineering, quantum mechanics, economics, solving differential equations etc.

***Powers of* i**

E.g. 1 Write down the value of these powers of i: i^2 i^3 i^4 i^5 .

N.B. i acts just like a normal constant and normal rules of arithmetic apply
We normally the letter z to represent a complex number.

Language

A complex number is one that can be written as $x + y\text{i}$ where x is said to be the ***real part*** and y is said to be the ***imaginary part***.

So if $z = 4 + 7\text{i}$ then Re(z) = 4 and Im(z) = 7

N.B. We do ***not*** say Im(z) = 7i
Two ***complex numbers*** are **equal** when their ***real and imaginary parts are equal***.

Operations with complex numbers

Addition and subtraction — add/subtract the real and imaginary parts separately.
Multiplication — use FOIL

E.g. 2 Let $z_1 = 3 - 4\text{i}$ and let $z_2 = 2 + 5\text{i}$. Find:
(a) $z_1 + z_2$ (b) $z_1 - z_2$ (c) $z_1 \times z_2$

E.g. 3 Let $z = a + b\text{i}$ where a and b are real numbers. If $z = 0$, find the values of a and b.

N.B. A complex number is zero when both the real and imaginary parts are zero.

Equating real and imaginary parts

Two complex numbers are equal when both the real and imaginary parts are equal

E.g. 4 Find the real numbers x and y such that $x + 4y + xy\text{i} = 12 - 16\text{i}$.

Working:

Equating real parts: $x + 4y = 12$

Equating imaginary parts: $xy = -16 \Rightarrow y = -\dfrac{16}{x}$

Substitute: $x + 4 \times \left(-\dfrac{16}{x}\right) = 12$

Multiply by x: $x^2 - 64 = 12x \Rightarrow x^2 - 12x - 64 = 0$

$(x - 16)(x + 4) = 0 \Rightarrow x = -4$ or $x = 16$

When $x = -4$, $y = -\dfrac{16}{-4} = 4$

When $x = 16$, $y = -\dfrac{16}{16} = -1$

Check in $x + 4y = 12$: $-4 + 4 \times 4 = 12$ True

$16 + 4 \times (-1) = 12$ True

Square rooting complex numbers

To find the square root of a complex number,

1. Equate $a + b\text{i}$ to the square root of the complex number.
2. Square both sides.
3. Equate real and imaginary parts to form 2 equations.
4. Solve the equations.

E.g. 5 Find the square root of $5 + 12\text{i}$.

Working

Let $z = a + b\text{i}$ be such that $z = \sqrt{5 + 12\text{i}}$, where a and b are real

i.e. $a + b\text{i} = \sqrt{5 + 12\text{i}}$

Squaring both sides gives $a^2 - b^2 + 2ab\text{i} = 5 + 12\text{i}$

Equating real and imaginary parts:

Re: $a^2 - b^2 = 5$

Im: $2ab = 12 \quad ab = 6 \quad b = \dfrac{6}{a}$

Substituting: $a^2 - \left(\dfrac{6}{a}\right)^2 = 5$

$a^4 - 36 = 5a^2$

$a^4 - 5a^2 - 36 = 0$

$(a^2 - 9)(a^2 + 4) = 0$

$a^2 = 9$ or $a^2 = -4$

$\therefore a = \pm 3$ since $a^2 = -4$ gives imaginary values

When $a = 3$, $b = 2$

When $a = -3$, $b = -2$

The square roots of $5 + 12\text{i}$ are $3 + 2\text{i}$ and $-3 - 2\text{i}$.

E.g. 6 Find the square root of $3 - 4\text{i}$.

Video: Real and imaginary numbers
Video: Complex number arithmetic
Video: Square rooting complex numbers

Solutions to Starter and E.g.s

Exercise
p113 4A Qu 1i, 2i, 3i, 4i, 5i, 6i, 7i, 8i, 9i, 10, 13-17 (not done matrices yet)

Summary
$\text{i} = \sqrt{-1}$

Two complex numbers are equal when both the real and imaginary parts are equal

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