Introduction to the Poisson Distribution

Starter

1. **(Review of last lesson)** In order to start a board game each player must throw at least one 6 with a pair of dice. Find the probability that for Jane to start she needs:
   (a) one throw
   (a) five throws
   (a) more than eight throws.

Notes

There is an infinite series such that:

$$e^{\lambda} = 1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \frac{\lambda^4}{4!} + \ldots + \frac{\lambda^r}{r!} + \ldots$$

Dividing by e^{λ} gives:

$$1 = e^{-\lambda} + \frac{e^{-\lambda}\lambda}{1!} + \frac{e^{-\lambda}\lambda^2}{2!} + \frac{e^{-\lambda}\lambda^3}{3!} + \frac{e^{-\lambda}\lambda^4}{4!} + \ldots + \frac{e^{-\lambda}\lambda^r}{r!} + \ldots$$

Since the sum of the terms is 1, we can use them to define a probability distribution where:

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} \textbf{ for } x = 0,1,2,3,4,\ldots$$

This is a useful discrete distribution called the Poisson distribution (named after Baron Simeon Denis Poisson 1781-1840).

N.B. By definition, $0! = 1$

Conditions for Poisson distribution

The Poisson distribution can be used for events which occur:

- At a constant rate (i.e. the mean number in an interval is proportional to the length of the interval)
- Independently of one another
- Randomly in space and time
- Singly in continuous space or time

For example, the number of:

- Calls to a switchboard
- Errors on a long piece of cloth
- Misprints on each page of a book
- Cars passing a point on a motorway in a time interval of 1 minute
- Radioactive particles emitted by a radioactive source in a time interval of 1 second

E.g. 1 Discuss with your partner which of the following could be modelled by a Poisson distribution. Give reasons for your answers.
(a) The number of misprints on this page in the first draft of this book.
(b) The number of pigs in a given square metre of a field 1 hour after feed was placed in a central trough.
(c) The number of pigs in a given square metre of a field 1 minute after feed was placed in a central trough.
(d) The amount of salt, in mg, contained in 1 cm^3 of water taken from a bucket immediately after a teaspoon of salt was added.
(e) The number of marathon runners passing the finishing post between 20 and 21 minutes after the winning of the race.

The parameter, λ

There is only one parameter, λ (lambda), in the formula for the Poisson distribution — it is the mean of the distribution. In fact, it is also variance of the distribution.

If $X \sim \text{Po}(\lambda)$ then

Mean: $\mu = E(X) = \lambda$

Variance: $\sigma^2 = \text{Var}(X) = \lambda$

Standard deviation: $\sigma = \sqrt{\lambda}$

The proof of these results is beyond the scope of the A level course.

Using your calculator

While calculating $P(X = 4)$ is relatively easy using the formula, it becomes more time-consuming when we need to calculate any of the following: $P(1 \leq X \leq 4)$, $P(X \leq 6)$ or $P(X \geq 8)$. The Classwiz calculator has a special function to enable use to calculate probabilities.

Menu >> 7: Distribution >> = >> "Down arrow" >> 2: Poisson PD or 3: Poisson CD >> 2: Variable

- 2: Poisson PD — use for $P(X = 4)$
- 3: Poisson CD — use for $P(1 \leq X \leq 4)$, $P(X \leq 6)$ or $P(X \geq 8)$

Poisson CD is the cumulative distribution and adds up all the probabilities until that point.

Video (Poisson PD): Finding Poisson PD with a calculator
Video (Poisson CD): Finding Poisson CD with a calculator
Video (list function): Using the list function

E.g. 1 Given that $X \sim \text{Po}(3.25)$, calculate, to 4 s.f. using the formula:

(a) $P(X = 3)$ (b) $P(X \leq 2)$ (c) $P(X \geq 2)$

Check your answers with the special function on your calculator.

Working: (a) ***By formula:*** $P(X = 3) = \dfrac{e^{-3.25} \times 3.25^3}{3!} = 0.2218$

Calculator: Use 2: Poisson PD $P(X = 3) = 0.2218$

E.g. 2 If $Y \sim \text{Po}(4.5)$, use the formula to find:

(a) $P(Y = 2)$ (b) $P(Y \leq 1)$ (c) $P(Y > 4)$ (d) $P(2 \leq Y \leq 6)$

Check your answers with the special function on your calculator.
For (d), use the special function on your calculator.

Video: Poisson distribution
Video: Poisson (changing the mean)
Video (Poisson PD): Finding Poisson PD with a calculator
Video (Poisson CD): Finding Poisson CD with a calculator
Video (list function): Using the list function

Solutions to Starter and E.g.s

Exercise

p41 3A 1ac, 2i

Summary

$$P(X = x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \text{ for } x = 0,1,2,3,4,...$$

If $X \sim \text{Po}(\lambda)$ then

Mean: $\mu = E(X) = \lambda$

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