Invariant points and invariant lines

Starter

1. A square undergoes a shear, $x-$axis invariant, mapping $(0, 1) \rightarrow (-4, 1)$. The point $(6, 2)$ is a vertex of the square before the shear. Find the new coordinates of the vertex.

2. Find the matrix which transforms $\begin{pmatrix} 4 \\ 3 \end{pmatrix} \rightarrow \begin{pmatrix} 9 \\ 10 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 5 \\ 6 \end{pmatrix}$.

3. Find the values of x and y such that:

   (a) $\begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ (b) $\begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

Notes

Invariant points

An invariant point is a point that is unaffected by a transformation.

Therefore, for matrix transformations this means: $\mathbf{M} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$.

Is it possible for a matrix transformation to have no invariant points?

No, the origin is always an invariant point as $\mathbf{M} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

E.g. 1 By letting $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, form and solve the simultaneous equations of $\mathbf{M} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ to show that $y = \left(\dfrac{a - c - 1}{d - b - 1} \right) x$ is the equation of invariant points.

E.g. 2 Using the equations formed from $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$, state what happens when:

(a) $a = 1$ (b) $d = 1$

Working: (a) $ax + by = x$

When $a = 1$: $x + by = x \Rightarrow y = 0$

Substituting $y = 0$ in $cx + dy = y$: $x = 0$

So the origin is the unique invariant point.

When $a \neq 1$, $d \neq 1$, $a - c \neq 1$ and $d - b \neq 1$, we have a line of invariant points passing through the origin.

Either we have a line of invariant points passing through the origin ***or*** the origin is the unique invariant point.

When solving equations, ***either***:

Simultaneous equations are ***identical*** $\Rightarrow$ ***invariant points lie on line passing through origin***

…or…

Simultaneous equations are ***different*** $\Rightarrow$ ***the origin is the unique invariant point***

E.g. 3 Find the invariant points under the transformation $\begin{pmatrix} 4 & 1 \\ 6 & 3 \end{pmatrix}$.

Working: $\begin{pmatrix} 4 & 1 \\ 6 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$4x + y = x \quad \Rightarrow \quad y = -3x$

$6x + 3y = y \quad \Rightarrow \quad y = -3x$

The equations are equal so the invariant points under the transformation $\begin{pmatrix} 4 & 1 \\ 6 & 3 \end{pmatrix}$ are all the points lying on the line $y = -3x$ i.e. all points of the form $(k, -3k)$.

E.g. 4 Find the invariant points under the transformation $\begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$.

Invariant lines

A line is an invariant line under a transformation if ***the image of a point on the line is also on the line***. It ***does not mean that every point on the line must map onto itself***.

N.B. Any line of invariant points is also an invariant line.

There are two options to consider: invariant lines through the origin and invariant lines not through the origin.

Invariant lines through the origin

An invariant line through the origin has the form $y = mx$.

Any point on this line is of the form (k, mk)

So if $y = mx$ is an invariant line: $\mathbf{M}\begin{pmatrix} k \\ mk \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

By expanding the matrix, equations involving x and y can be formed.

By rearranging, k can be eliminated to form an equation where $y = f(m)x$.

Since $y = mx$ and $y = f(m)x$ must be the same line we can solve the equation $m = f(m)$ to find the values of m that gives invariant points.

E.g. 5 Find the equation of any invariant lines through the origin of the transformation whose matrix is $\begin{pmatrix} 2 & 3 \\ 0 & -1 \end{pmatrix}$.

Working: Any point on the invariant line has coordinates of the form (k, mk)

$\begin{pmatrix} 2 & 3 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} k \\ mk \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$x = 2k + 3mk \quad \Rightarrow \quad x = k(2 + 3m) \quad \Rightarrow \quad k = \dfrac{x}{2 + 3m}$

$y = -mk$

Substituting $k = \dfrac{x}{2 + 3m}$: $\quad y = \left(\dfrac{-m}{2 + 3m}\right)x$

Since the gradient must be the same as $y = mx$: $\quad m = \dfrac{-m}{2 + 3m}$

$3m^2 + 3m = 0 \quad \Rightarrow \quad m(m + 1) = 0 \quad \Rightarrow \quad m = 0$ & $m = -1$

So $y = 0$ and $y = -x$ are the invariant lines passing through the origin.

E.g. 6 Find the invariant lines of the matrix $\begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$ which pass through the origin.

Invariant lines not passing through the origin

An invariant line not through the origin has the form $y = mx + c$, where $c \neq 0$.
Any point on this line is of the form $(k, mk + c)$

So if $y = mx + c$ is an invariant line: $\mathbf{M}\begin{pmatrix} k \\ mk + c \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

Again expand the matrix and form equations involving x and y.
By rearranging, k can be eliminated to form an equation where $y = f(m)x + g(m)c$.
The lines $y = mx + c$ and $y = f(m)x + g(m)c$ must be the same.
By equating coefficients of x and y we get: $m = f(m)$ and $1 = g(m)$
Solve the equation $m = f(m)$ to find values of m.
Check whether these $m-$values satisfy $1 = g(m)$:

If they do, $y = mx + c$ is an invariant line
If not, $1 = g(m)$ is only satisfied when $c = 0$ so $y = mx$ is an invariant line.

E.g. 7 Find the invariant lines of the matrix $\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$.

Working: Any point on the invariant line is of the form $(k, mk + c)$.

$$\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} k \\ mk + c \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$x = 2k + mk + c \quad \Rightarrow \quad x = k(2 + m) + c$$

$$\Rightarrow \quad k = \frac{1}{2 + m}x - \frac{1}{2 + m}c$$

$$y = 2k + 3mk + 3c \quad \Rightarrow \quad y = k(2 + 3m) + 3c$$

Substituting $k = \frac{1}{2 + m}x - \frac{1}{2 + m}c$:

$$y = \left(\frac{1}{2 + m}x - \frac{1}{2 + m}c\right)(2 + 3m) + 3c$$

$$y = \left(\frac{2 + 3m}{2 + m}\right)x + \left(3 - \frac{2 + 3m}{2 + m}\right)c$$

This image point must lie on the line $y = mx + c$, so equating coefficients of x and c:

$$x: \quad m = \frac{2 + 3m}{2 + m} \quad \Rightarrow \quad m^2 - m - 2 = 0$$

$$\Rightarrow \quad (m - 2)(m + 1) = 0$$

$$\Rightarrow \quad m = 2 \text{ or } m = -1$$

Now substitute into the equation formed by equating coefficients of c to see if it satisfies the equation.

$$c: \quad 3 - \frac{2 + 3m}{2 + m} = 1$$

$$m = 2: \quad 3 - \frac{2 + 3 \times 2}{2 + 2} = 3 - \frac{8}{4} = 1 \quad ✓$$

So $y = 2x + c$ is an invariant line.

$$m = -1: \quad 3 - \frac{2 + 3 \times (-1)}{2 + (-1)} = 3 - \frac{-1}{1} \neq 1$$

$\therefore \left(3 - \frac{2 + 3m}{2 + m}\right)c = c$ is only true when $c = 0$.

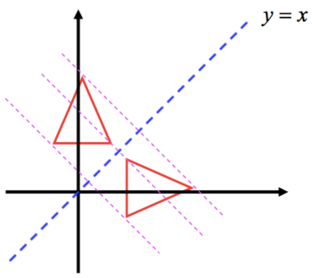
So $y = -x$ is the other invariant line.

E.g. 8 Find any invariant lines of the matrix $\begin{pmatrix} 4 & 3 \\ -3 & -2 \end{pmatrix}$

Types of transformations and invariant lines

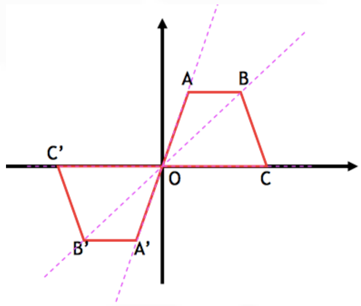
Reflections

Any line that is perpendicular to the mirror line will be an invariant line.



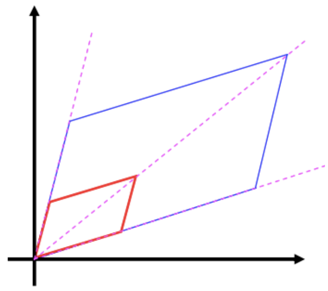
Rotations

Only rotations of 180° about the origin have an invariant line. In such cases $y = kx$ is invariant



Enlargements

Any line passing through the origin is an invariant line i.e. $y = kx$



Video A: Invariant points and lines
Video B: Invariant points and lines

Solutions to Starter and E.g.s

Exercise

p87 3D Qu 1ac, 2ace, 3-5

Summary

An invariant point is a point that is unaffected by a transformation i.e. $\mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

Invariant lines through the origin: $\mathbf{M}\begin{pmatrix} k \\ mk \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

Invariant lines not passing through the origin: $\mathbf{M}\begin{pmatrix} k \\ mk + c \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$