Inverse of 2 by 2 Matrices

Starter

1. Find the values of k such that the matrix $\mathbf{M}$ is singular, where $\mathbf{M} = \begin{pmatrix} 2k & 18 \\ 4 & k-3.5 \end{pmatrix}$.

2. The 2 by 2 identity matrix, $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Find:

   (a) $\begin{pmatrix} 5 & -8 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

3. Let matrix $\mathbf{X} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

   (a) Find the values of a, b, c and d such that $\mathbf{X}\begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

   (b) Calculate the product $\begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix}\mathbf{X}$ using the values found in (a).

4. Let matrix $\mathbf{P} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

   Find the values of a, b, c and d such that $\mathbf{P}\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Notes

From the starter we can see that identity matrices have the same property as the number 1 in multiplication i.e. it leaves a matrix untouched after multiplication

Multiplication: $5 \times 1 = 1 \times 5 = 5$ 1 is the identity element in multiplication

Matrices: $\mathbf{A}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{A} = \mathbf{A}$ $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity matrix for 2 by 2

Notation

The inverse matrix of $\mathbf{A}$ is denoted by $\mathbf{A}^{-1}$.

Inverse matrix

In multiplication, a number can combine with its reciprocal to give the identity element.

Multiplication: $5 \times \frac{1}{5} = 1$ ***or*** $\frac{1}{5} \times 5 = 1$

With matrices, a matrix multiplies its inverse to give the identity matrix.the inverse matrix of $\mathbf{A}$ is

Matrices: $\mathbf{A}\mathbf{A}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ***or*** $\mathbf{A}^{-1}\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

2 by 2 matrices

In question 3 of the starter we used simultaneous equations to find the inverse of the matrix, but we need a quicker method.

E.g. 1 Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and let $\mathbf{B} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

(a) Find the product $\mathbf{AB}$.

(b) State which needs to be done to $\mathbf{AB}$ make it equal to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(c) Hence state the inverse matrix of $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Include det $\mathbf{A}$ in your answer.

The inverse of the 2 by 2 matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is:

$$\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad \textit{...or...} \quad \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Do all 2 by 2 matrices have an inverse?
Question 4 of the starter showed that not every matrix has an inverse. In the formula for the inverse, we divide by det $\mathbf{A}$. Therefore, if det $\mathbf{A} = 0$, the inverse matrix does not exist.

i.e. ***singular matrices do not have an inverse***

When asked to find the inverse of a matrix, it is best to calculate its determinant first.

E.g. 2 If it exists, find the inverse of the matrix:

(a) $\begin{pmatrix} 4 & 9 \\ 3 & 7 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & -6 \\ -3 & 9 \end{pmatrix}$ (c) $\begin{pmatrix} 5 & -3 \\ 10 & -5 \end{pmatrix}$

Working: (a)

$$\left|\begin{pmatrix} 4 & 9 \\ 3 & 7 \end{pmatrix}\right| = 4 \times 7 - 3 \times 9 = 1$$

$$\begin{pmatrix} 4 & 9 \\ 3 & 7 \end{pmatrix}^{-1} = \frac{1}{1}\begin{pmatrix} 7 & -9 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} 7 & -9 \\ -3 & 4 \end{pmatrix}$$

Inverse of a product
What is the inverse of the product of matrices $\mathbf{AB}$?
Let $\mathbf{X}$ be the inverse of $\mathbf{AB}$.
$(\mathbf{AB})\mathbf{X} = \mathbf{I}$
By associativity: $\mathbf{A}(\mathbf{BX}) = \mathbf{I}$
Pre-multiply by $\mathbf{A}^{-1}$: $\mathbf{A}^{-1}\mathbf{A}(\mathbf{BX}) = \mathbf{A}^{-1}\mathbf{I}$ so $\mathbf{I}(\mathbf{BX}) = \mathbf{A}^{-1} \Rightarrow \mathbf{BX} = \mathbf{A}^{-1}$
Pre-multiply by $\mathbf{B}^{-1}$: $\mathbf{B}^{-1}\mathbf{BX} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ so $\mathbf{IX} = \mathbf{B}^{-1}\mathbf{A}^{-1} \Rightarrow \mathbf{X} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
Therefore the inverse of $\mathbf{AB}$ is $\mathbf{B}^{-1}\mathbf{A}^{-1}$ i.e. $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

N.B. Since matrix multiplication is not commutative, you need to specify whether to ***pre-*** or ***post-multiply*** by a matrix.

E.g. 3 Let $\mathbf{X} = \begin{pmatrix} 4 & 7 \\ 2 & 3 \end{pmatrix}$ and $\mathbf{Y} = \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}$. Calculate the inverse of $\mathbf{YX}$.

E.g. 4 Solve these matrix equations:

(a) $\begin{pmatrix}1 & 3\\ 2 & 4\end{pmatrix}\mathbf{X} = \begin{pmatrix}-2 & 9\\ -2 & 14\end{pmatrix}$ (b) $\mathbf{X}\begin{pmatrix}1 & 3\\ 2 & 4\end{pmatrix} = \begin{pmatrix}-2 & 9\\ -2 & 14\end{pmatrix}$

Hint: Remember to specify whether you are pre- or post-multiplying by a matrix.

Working: (a) Pre-multiply by $\begin{pmatrix}1 & 3\\ 2 & 4\end{pmatrix}^{-1}$:

$$\begin{pmatrix}1 & 3\\ 2 & 4\end{pmatrix}^{-1}\begin{pmatrix}1 & 3\\ 2 & 4\end{pmatrix}\mathbf{X} = \begin{pmatrix}1 & 3\\ 2 & 4\end{pmatrix}^{-1}\begin{pmatrix}-2 & 9\\ -2 & 14\end{pmatrix}$$

But $\begin{pmatrix}1 & 3\\ 2 & 4\end{pmatrix}^{-1}\begin{pmatrix}1 & 3\\ 2 & 4\end{pmatrix} = \mathbf{I}$ and $\mathbf{IX} = \mathbf{X}$.

So $\mathbf{X} = \begin{pmatrix}1 & 3\\ 2 & 4\end{pmatrix}^{-1}\begin{pmatrix}-2 & 9\\ -2 & 14\end{pmatrix}$

$$\begin{pmatrix}1 & 3\\ 2 & 4\end{pmatrix}^{-1} = \frac{1}{-2}\begin{pmatrix}4 & -3\\ -2 & 1\end{pmatrix}$$

$$\therefore \mathbf{X} = \frac{1}{-2}\begin{pmatrix}4 & -3\\ -2 & 1\end{pmatrix}\begin{pmatrix}-2 & 9\\ -2 & 14\end{pmatrix} = \begin{pmatrix}1 & 3\\ -1 & 2\end{pmatrix}$$

Video: Inverse of 2 by 2 matrices

Video: Transposed and symmetric matrices

Inverse of 2 by 2 matrices EQ

Solutions to Starter and E.g.s

Exercise

p22 1D Qu 1i, 2i, 3i, 4-10

Summary

The inverse of the 2 by 2 matrix $\mathbf{A} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$ is:

$$\mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}$$ ***...or...*** $$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}$$

Inverse of a product: $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$