Linear Simultaneous Equations

Starter

1. Find the inverse of the matrix $\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ -2 & -2 & 0 \end{pmatrix}$ by hand.

2. Using your calculator, solve for $\mathbf{X}$ the equation $\mathbf{X}\begin{pmatrix} 3 & -1 & 5 \\ 2 & 4 & -2 \\ 1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 18 & 29 & -3 \\ 7 & 0 & 12 \\ 7 & 0 & 10 \end{pmatrix}$.

3. The system of simultaneous equations

$$\begin{aligned} 2x + 3y &= 1 \\ 4x - \ \ y &= 9 \end{aligned}$$

can be expressed in matrix form $\begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 9 \end{pmatrix}$

Solve the equations by way of an inverse matrix.

Notes

Simultaneous equations of the form:

$$\begin{aligned} a_1x + b_1y &= d_1 \\ a_2x + b_2y &= d_2 \end{aligned}$$

…can be re-written in matrix form as:

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$$

By ***pre-multiplying*** by $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}^{-1}$ we get:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}^{-1}\begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$$

Similarly 3 equations and 3 unknowns of the form:

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

…can be written as:

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

…and can be solved…

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}^{-1}\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

In short, if $\mathbf{AX} = \mathbf{B}$ then $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$.

This gives a unique solution when $\mathbf{A}^{-1}$ exists i.e. when $\Delta = \det \mathbf{A} \neq 0$
i.e. unique solution exists when the ***matrix of coefficients*** is not singular.

Non-unique solutions in 2-D

There are two options in $2-$dimensions if the two lines do not meet in a unique point:

1. Infinitely many solutions (i.e. the lines are coincident $\Rightarrow$ the equations are multiples of one another)
2. No solutions (i.e. the lines are parallel $\Rightarrow$ the coefficients are in the same ratio, but the constant is not)

$3-$dimensions will be considered in the A2 course.

E.g. 1 Solve these equations using matrices:
(a) $x + 2y = 2,\ 3x + 4y = 8$ (b) $x + 2y = 1,\ 3x + 5y = 4$

Working: (a) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$

Pre-multiply by $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1}$: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1}\begin{pmatrix} 2 \\ 8 \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 8 \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$

$x = 4,\ y = -1$

E.g. 2 Find the values of k for which these systems of equations have a unique solution:
(a) $x + 2y = 2,\ 3x + ky = 5$ (b) $3x - 4y = 7,\ kx - 3y = 8$

Working: (a) The matrix of coefficients is $\begin{pmatrix} 1 & 2 \\ 3 & k \end{pmatrix}$

For a unique solution $\begin{vmatrix} 1 & 2 \\ 3 & k \end{vmatrix} \neq 0$

$k - 6 \neq 0$

The equations have a unique solution when $k \neq 6$.

E.g. 3 Find the value of k for which these systems of equations do not have a unique solution. State whether there are no solutions or an infinite number of solutions and give a geometrical explanation.
(a) $x + y = 2,\ -x + ky = -2$ (b) $2x + 3y = 2,\ -6x + ky = -8$

Working: (a) The matrix of coefficients is $\begin{pmatrix} 1 & 1 \\ -1 & k \end{pmatrix}$.

For no unique solution $\begin{vmatrix} 1 & 1 \\ -1 & k \end{vmatrix} = 0$

$k + 1 = 0$

The equations have no unique solution when $k = -1$.
When $k = -1$ the equations are multiples of each other i.e. they are coincident (the same line) so there are infinite solutions.

E.g. 4 By using matrices, solve these systems of linear equations. You may use your calculator to find the inverse and perform matrix multiplication

(a) $2x - y - z = 4, x + 2y = 10, y - z = 2$

(b) $x + y - z = 3, 3x - 2y - z = 1, 2x + 3y - z = 9$

Working: (a) $\begin{pmatrix} 2 & -1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 10 \\ 2 \end{pmatrix}$

Pre-multiply by $\begin{pmatrix} 2 & -1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix}^{-1}$:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 4 \\ 10 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix}$$

$x = 4, y = 3, z = 1$

Video: Solving simultaneous equations using matrices

Solutions to Starter and E.g.s

Exercise

p68 3A Qu 1i, 2i, 3-9

Summary

$\mathbf{AX} = \mathbf{B} \quad \Rightarrow \quad \mathbf{X} = \mathbf{A}^{-1}\mathbf{B}.$

$a_1x + b_1y = d_1$

$a_2x + b_2y = d_2$

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}^{-1}\begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$$

$a_1x + b_1y + c_1z = d_1$

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}^{-1}\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

This gives a unique solution when $\mathbf{A}^{-1}$ exists i.e. when $\Delta = \det \mathbf{A} \neq 0$

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There are two options in 2-dimensions if the two lines do not meet in a unique point:

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