Matrices as linear transformations

Starter

1 Find the value of k for which the systems of equations $4x + 3y = 2$ and $kx - 6y = -4$ do not have a unique solution. State whether there are no solutions or an infinite number of solutions and give a geometrical explanation.

2. By using matrices, solve this systems of linear equations:

$$2x - 3y + 4z = 7 \qquad x - 2y + 3z = 5 \qquad 3x - 5y + 2z = 2$$

You may use your calculator to find the inverse and perform matrix multiplication

3. Find: (a) $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

4. By viewing question 1 as a matrix acting on the point with coordinates $(2, 1)$ and transforming it to the coordinates of your answer, describe the transformation (rotation or reflection) that is happening in each case. If you need, sketch a graph to help you.

Notes

The best way to understand the transformation of a matrix is to consider its effect on the unit square: $(0, 0)$, $(1, 0)$, $(1, 1)$ and $(0, 1)$. In reality $(0, 0)$ will always map to $(0, 0)$ so we just need to consider the other three points.

Let $A(1, 0)$, $B(1, 1)$ and $C(0, 1)$ and then label the images A', B' and C' respectively.

i.e. $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$ so $A\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to A'\begin{pmatrix} a \\ c \end{pmatrix}$ and repeat for B and C.

E.g. 1 By considering the effect of the matrix on the unit square, describe the transformation under:

(a) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

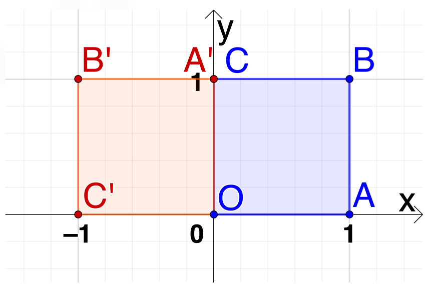
Hint: Write the points as column vectors.

Working: (a)

A: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

B: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

C: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$



Rotation $90°$ anti-clockwise about the origin

For rotations, if the direction is not given, the convention is that a ***positive angle*** means an ***anti-clockwise*** direction.

Effect of the determinant

The determinant of the matrix gives the area factor.
If the determinant is less than zero, the orientation of the shape is reversed.

det $\mathbf{M} = 1 \quad \Rightarrow$ rotation
det $\mathbf{M} = -1 \quad \Rightarrow$ reflection
det $\mathbf{M} = k^2 \quad \Rightarrow$ enlargement, factor k

N.B. An enlargement in only one direction is called a stretch.
Since a matrix transformation never moves the origin, translations cannot be represented by a matrix.

Finding the matrix based on a transformation

Again we use the unit square as the basis for finding the transformation matrix. Consider the effect on $(1,\ 0)$ and $(0,\ 1)$ first and confirm with $(1,\ 1)$.

If $\begin{pmatrix}1\\0\end{pmatrix} \to \begin{pmatrix}x_1\\y_1\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix} \to \begin{pmatrix}x_2\\y_2\end{pmatrix}$ then the matrix is $\begin{pmatrix}x_1 & x_2\\y_1 & y_2\end{pmatrix}$

N.B. Draw a diagram if you cannot picture to where the vertices of the unit square go.
Write the points as column vectors.

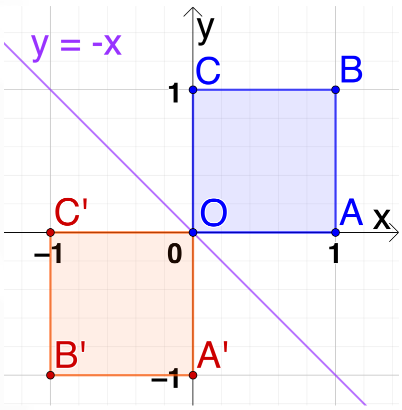
E.g. 2 Find the matrix that causes a reflection in the line $y = -x$.

Working: $\begin{pmatrix}1\\0\end{pmatrix} \to \begin{pmatrix}0\\-1\end{pmatrix} \quad \begin{pmatrix}0\\1\end{pmatrix} \to \begin{pmatrix}-1\\0\end{pmatrix}$

so the matrix is $\begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}$

Confirm using $\begin{pmatrix}1\\1\end{pmatrix}$:

$\begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-1\\-1\end{pmatrix}$ ✓



E.g. 3 Find the matrix that causes a reflection in the line $y-$axis.

E.g. 4 The matrix $\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$ causes a rotation of $90°$ ***anti-clockwise*** about the origin. Using this matrix, find the matrix that causes a rotation of $90°$ ***clockwise*** about the origin.

Inverse matrices and transformations

The inverse matrix, $\mathbf{M}^{-1}$, reverses the transformation of the original matrix $\mathbf{M}$.

Successive transformation

Consider two transformation caused by the matrices $\mathbf{A}$ and $\mathbf{B}$. The matrix that multiples the coordinates, $\mathbf{X}$, first is the first to act upon the shape:
$\mathbf{ABX}$ - transformation $\mathbf{B}$ following by $\mathbf{A}$
$\mathbf{BAX}$ - transformation $\mathbf{A}$ followed by $\mathbf{B}$

E.g. 5 Find the matrix that causes an enlargement by a length factor of 2 followed by a rotation of $90°$anti-clockwise about the origin.

Working: Enlargement by a length factor of 2 is $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ ***1st matrix to multiply***

Rotation, $90°$anti-clockwise about origin $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ ***2nd matrix to multiply***

so $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\mathbf{X} = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}\mathbf{X}$

The required matrix is $\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$

E.g. 6 Given that $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ causes a reflection in the $y-$axis and $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ causes a rotation of $180°$, centre origin, find the single matrix that causes a reflection in the $y-$axis followed by a rotation of $180°$, centre the origin.

Video: Matrix reflections
Video: Matrix enlargments
Video: Inverse marices and transformations

Solutions to Starter and E.g.s

Exercise
p76 3B Qu 1i, 2i, 3i, 4i, 5i, 6-11

Summary
To find the transformation caused by a matrix consider its effect on the unit square: $(1, 0)$, $(1, 1)$ and $(0, 1)$.

Effect of the determinant:
The determinant of the matrix gives the area factor.
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Finding the matrix based on a transformation:
Again we use the unit square as the basis for finding the transformation matrix. Consider the effect on $(1, 0)$ and $(0, 1)$ first and confirm with $(1, 1)$.

Inverse matrices and transformations:
The inverse matrix, $\mathbf{M}^{-1}$, reverses the transformation of the original matrix $\mathbf{M}$.

Successive transformations:
$\mathbf{ABX}$ - transformation $\mathbf{B}$ following by $\mathbf{A}$
$\mathbf{BAX}$ - transformation $\mathbf{A}$ followed by $\mathbf{B}$