Matrix Multiplication

Starter

1. For the matrix $\mathbf{A} = \begin{pmatrix} 3 & 5 & -2 \\ 4 & -1 & -3 \\ 8 & 6 & -7 \end{pmatrix}$ state the number that corresponds to these elements:

(a) a_{21} (b) a_{32} (c) a_{13}

Notes

When multiplying matrices the ***rows*** of the 1st matrix ***get multiplied by*** the ***columns*** of the 2nd matrix.

E.g. Find $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} e & f \\ g & h \end{pmatrix}$.

Working:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & \\ & \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} & af+bh \\ & \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} & \\ ce+dg & \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} & \\ & cf+dh \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

Let $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ be the result of the product of two matrices.

a_{13} is the element in the ***1st row***, ***3rd column***
So a_{13} is the product of the ***1st row*** of the first matrix by the ***3rd column*** of the second matrix.

a_{32} is the element in the ***3rd row***, ***2nd column***
So a_{32} is the product of the ***3rd row*** of the first matrix by the ***2nd column*** of the second matrix.

E.g. 1 Let $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -5 & 6 \\ 7 & 8 \end{pmatrix}$. Find:

(a) $\mathbf{AB}$ (b) $\mathbf{BA}$ (c) $\mathbf{A}^2$.

Working: (a)

$$\mathbf{AB} = \begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix} \times \begin{pmatrix} -5 & 6 \\ 7 & 8 \end{pmatrix}$$
$$= \begin{pmatrix} 1 \times (-5) + 2 \times 7 & 1 \times 6 + 2 \times 8 \\ (-3) \times (-5) + 4 \times 7 & (-3) \times 6 + 4 \times 8 \end{pmatrix}$$
$$= \begin{pmatrix} 9 & 22 \\ 43 & 14 \end{pmatrix}$$

Square matrices of the same size can always be multiplied together.

Deciding whether 2 matrices can multiply

Square matrices of the same size can always be multiplied together.

What happens when the two matrices are not same-size squares matrices?

Each element in a ***row*** of the first matrix must have an element in a ***column*** from the second matrix to multiply by.

Consider $\begin{pmatrix} 2 & -3 & 1 \\ -5 & 2 & -2 \end{pmatrix} \times \begin{pmatrix} 7 & 8 \\ 9 & 10 \end{pmatrix}$.

Each row of the first matrix has 3 elements but each column of the second matrix has only 2 elements. When we try to multiply the 1st row of the first matrix by the 1st column of the second matrix, the element 1 does not have an element to multiply by. Therefore, we cannot find the product of these two matrices.

The dimensions of the first matrix are 2 by 3.
The dimensions of the second matrix are 2 by 2.

When we try and multiply them together: 2 by 3×2 by 2

Since the middle numbers are not equal we cannot multiply the two matrices. Think of it as the ***domino rule***.

Let matrix $\mathbf{A}$ have dimensions m by n and let matrix $\mathbf{B}$ have dimensions p by q.

$\mathbf{AB}$: m by $n \times p$ by q This is possible iff $n = p$. (iff ≡ "if and only if")

$\mathbf{BA}$: p by $q \times m$ by n This is possible iff $q = m$

E.g. 1 Let $\mathbf{A} = \begin{pmatrix} 2 & -3 & 1 \\ -5 & 2 & -2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \\ 3 & 6 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 7 & 8 \\ 9 & 10 \end{pmatrix}$.

Decide whether it is possible to find these products:

(a) $\mathbf{AB}$ (b) $\mathbf{AC}$ (c) $\mathbf{BA}$ (d) $\mathbf{B}^2$

Working: (a) $\mathbf{AB}$: 2 by 3×3 by 2 Possible

N.B. A matrix can only multiply itself when it is a square matrix.

E.g. 2 Using the matrices from ***E.g. 2***, find $\mathbf{AB}$. Write down the dimensions of the resultant matrix.

$\mathbf{AB}$: 2 by 3×3 by 2 The dimensions of the resultant matrix are 2 by 2 i.e. the dimensions of the resultant are given by the outside numbers.

Domino rule

Let matrix $\mathbf{A}$ have dimensions m by n and let matrix $\mathbf{B}$ have dimensions p by q.

$\mathbf{AB}$: m by $n \times p$ by q

The product $\mathbf{AB}$ can be found iff $n = p$

The resultant matrix has dimensions m by q

E.g. 3 Let $\mathbf{A} = \begin{pmatrix} 2 & -3 & 1 \\ -5 & 2 & -2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \\ 3 & 6 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 7 & 8 \\ 9 & 10 \end{pmatrix}$.

State the dimensions of the resultant matrix of these products:

(a) $\mathbf{BA}$ (b) $\mathbf{CA}$

Working: (a) $\mathbf{BA}$: 3 by 2×2 by 3 Resultant has dimensions 3 by 3

N.B. ***Before multiplying*** matrices make sure you ***know the dimensions of the resultant matrix***.

$$\mathbf{BA} = \begin{pmatrix} -13 & 3 & -5 \\ -16 & 2 & -6 \\ -24 & 3 & -9 \end{pmatrix}$$

AC = not possible

Commutativity

Matrix multiplication is not generally commutative so usually $\mathbf{AB} \neq \mathbf{BA}$ (this is different to normal multiplication).

Associativity

Matrix multiplication is associative so $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$

E.g. 4 Let $\mathbf{P} = \begin{pmatrix} -1 & 2 & 3 \\ 4 & -5 & 6 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 7 & 8 \\ 9 & -10 \\ 11 & 12 \end{pmatrix}$. Find:

(a) $\mathbf{PQ}$ (b) $\mathbf{QP}$.

Working: (a) $\mathbf{PQ}$: 2 by 3×3 by 2 $\Rightarrow$ Resultant matrix is 2 by 2

$$\mathbf{PQ} = \begin{pmatrix} -1 & 2 & 3 \\ 4 & -5 & 6 \end{pmatrix}\begin{pmatrix} 7 & 8 \\ 9 & -10 \\ 11 & 12 \end{pmatrix}$$

$$= \begin{pmatrix} (-1)\times 7 + 2\times 9 + 3\times 11 & (-1)\times 8 + 2\times(-10) + 3\times 12 \\ 4\times 7 + (-5)\times 9 + 6\times 11 & 4\times 8 + (-5)\times(-10) + 6\times 12 \end{pmatrix}$$

$$= \begin{pmatrix} 44 & 8 \\ 49 & 154 \end{pmatrix}$$

Video: Matrix multiplication

Solutions to Starter and E.g.s

Exercise
p12 1B Qu 1i, 2i, 3i, 4i, 5-12

Summary
When multiplying matrices the ***rows*** of the 1st matrix ***get multiplied by*** the ***columns*** of the 2nd matrix.
a_{32} is the element in the ***3rd row***, ***2nd column***
So a_{32} is the product of the ***3rd row*** of the first matrix by the ***2nd column*** of the second matrix.

Domino rule:
Let matrix $\mathbf{A}$ have dimensions m by n and let matrix $\mathbf{B}$ have dimensions p by q.
$\mathbf{AB}$: m by $n \times p$ by q
The product $\mathbf{AB}$ can be found iff $n = p$
The resultant matrix has dimensions m by q