Matrix arithmetic

Notes

A matrix is a rectangular array of numbers, called elements, surrounding by curved brackets.

For example, $\mathbf{A} = \begin{pmatrix} 2 & -6 & 7 \\ -5 & 8 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 \\ 6 \\ -2 \\ 9 \end{pmatrix}$

A matrix $\mathbf{M}$ with m rows and n columns is called an m **by** n matrix or $m \times n$ matrix.

Matrix $\mathbf{A}$ has 2 rows and 3 columns so is a 2 by 3 matrix ***or*** 2×3 matrix.
Matrix $\mathbf{B}$ has 4 rows and 1 column so is a 4 by 1 matrix ***or*** 4×1 matrix.

If $m = n$, $\mathbf{M}$ is said to be a square matrix. ***E.g.*** $\begin{pmatrix} 5 & -2 \\ 1 & 4 \end{pmatrix}$

A matrix with n rows and 1 column is called a column matrix. The matrix $\mathbf{B}$ above is a column matrix.

A matrix with 1 row is called a row matrix. ***E.g.*** $\begin{pmatrix} 8 & -3 & 7 \end{pmatrix}$

Addition and subtraction

To add or subtract two matrices, they must be exactly the ***same dimensions***.
When two matrices are the same dimensions, we add them by adding corresponding elements.

E.g. $$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 10 & 12 \end{pmatrix}$$

The addition of matrices is:

commutative i.e. $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$

and

associative i.e. $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$.

Zero matrix, $\mathbf{Z}$

A matrix is said to be a ***zero*** (or null) ***matrix*** when ***every element is zero***. ***E.g.*** $\mathbf{Z}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Identity matrix, $\mathbf{I}$

The identity matrix, $\mathbf{I}$, is a square matrix with 1 as the elements of the lead diagonal and zeros everywhere else.

E.g. $\mathbf{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\qquad \mathbf{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Equal matrices

Two matrices are equal if they are the ***same dimensions*** and ***every corresponding element is the same***.

E.g. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \neq \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ $\qquad \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \end{pmatrix}$

Scalar multiples

When multiplying a matrix by a scalar, ***each element*** of the matrix ***is multiplied by the scalar***.

E.g. $k\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$

Transpose of a matrix

The transpose of a matrix $\mathbf{A}$ is a new matrix, denoted by $\mathbf{A}^T$, whereby the rows and columns of the matrix have been swapped over. When a matrix is a non-square matrix, this means changing the dimensions.

E.g. $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ $\qquad$ $\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}^T = \begin{pmatrix} a & c & e \\ b & d & f \end{pmatrix}$

N.B. For square matrices that are transposed, elements on the ***lead diagonal*** remain in the same position.

Denoting individual elements

Individual elements of the matrix $\mathbf{A}$ can be denoted a_{ij} where i is the row and j is the column.

i.e. $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$

N.B. For matrices, it is always "rows by columns".

E.g. 1 Let $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix}$. Calculate the matrices:

(a) $\mathbf{A} + \mathbf{B}$ (b) $\mathbf{A} - \mathbf{B}$ (c) $3\mathbf{A} + 2\mathbf{B}$ (d) $4\mathbf{A} - 3\mathbf{B}$

Working: (a) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 7 & 3 \end{pmatrix}$

E.g. 2 Solve for $\mathbf{X}$ the matrix equation $2\mathbf{X} + 3\mathbf{A} = 4\mathbf{X} - 3\mathbf{B}$. What do you need to assume for your answer to be correct?

E.g. 3 Write down the transpose of the following matrices:

(a) $\begin{pmatrix} 6 & -2 & 5 & 3 \\ 0 & 4 & -7 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 8 & 3 \\ -5 & 7 \end{pmatrix}$ (c) $\begin{pmatrix} 2 \\ -7 \\ 3 \end{pmatrix}$

Working: (a) $\begin{pmatrix} 6 & -2 & 5 & 3 \\ 0 & 4 & -7 & 1 \end{pmatrix}^T = \begin{pmatrix} 6 & 0 \\ -2 & 4 \\ 5 & -7 \\ 3 & 1 \end{pmatrix}$

N.B. A transposed column matrix becomes a row matrix, and vice versa.

E.g. 4 For the matrix $\mathbf{A} = \begin{pmatrix} 4 & 6 & -1 \\ 5 & 0 & -2 \\ 9 & 7 & -8 \end{pmatrix}$ state the number that corresponds to these elements:

(a) a_{13} (b) a_{32} (c) a_{23}

Working: (a) a_{13} is the element in the **1st** row, **3rd** column
$a_{13} = -1$

Video: Dimensions of a matrix
Video: Matrix arithmetic

Solutions to Starter and E.g.s

Exercise
p6 1A Qu 1i, 2i, 3i, 4i, 5i, 6i, 7-9

Summary
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If $m = n$, $\mathbf{M}$ is said to be a square matrix.
A matrix with n rows and 1 column is called a column matrix.
To ***add*** or ***subtract*** two matrices, they must be exactly the ***same size***.
When two matrices are the same size, we add them by ***adding corresponding elements***.
A matrix is said to be a ***zero*** (or null) ***matrix*** when ***every element is zero***.
Two matrices are ***equal*** if they are the same size and ***every corresponding element is the same***.
When ***multiplying a matrix by a scalar***, ***each element*** of the matrix ***is multiplied by the scalar***.
The ***transpose of a matrix*** $\mathbf{A}$ is a new matrix, denoted by $\mathbf{A}^T$, whereby the rows and columns of the matrix have been swapped over. When a matrix is a non-square matrix, this means changing the dimensions.