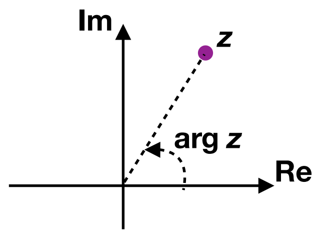
Modulus-Argument Form of a Complex Number

Starter

1. **(Review of last lesson)**
   Convert 240° to an angle in radians, expressing your answer in terms of π.

2. **(Review of last lesson)** Find the modulus of $4 + 2\text{i}$.

N.B. The argument of a complex number, z, is the ***angle*** that the line ***between the origin and z makes with the positive $x-$axis***, ***measured anti-clockwise***. It is denoted arg z and is given ***in radians***.



3. Calculate the argument of the complex numbers:
   (a) $1 + \text{i}$ (b) 4i (c) $-\sqrt{3} + \text{i}$
   Hint: use an Argand diagram to help you.

4. You are given the modulus and argument of a complex number. Express the complex number in the form $x + y\text{i}$.
   (a) Modulus = 6, argument = $\dfrac{\pi}{3}$ (b) Modulus = 2, argument = $\dfrac{3\pi}{2}$
   Hint: draw an Argand diagram to help.

Notes

The modulus-argument form of a complex number, z, consists of the modulus, r, which is the distance to the origin, and the argument θ, which is the angle the line Oz makes with the ***positive $x-$axis***, ***measured anti-clockwise***.

N.B. $r \geq 0$
The angle θ can take any real value but the principal argument, denoted by Arg z, is defined as $0 \leq \theta < 2\pi$ ***or*** $-\pi < \theta \leq \pi$

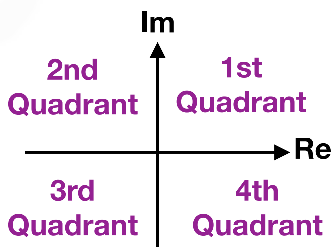
There are two forms of a complex number:

Cartesian form $x + y\text{i}$
Modulus-argument form $[r,\ \theta]$ — we will see that this notation is rarely used

Converting between Cartesian and modulus-argument forms
Cartesian to modulus-argument form

For the complex number $z = x + y\text{i}$

The modulus is given by $r = \sqrt{x^2 + y^2}$.



To find the argument:

1. Calculate $\tan^{-1}\left|\dfrac{y}{x}\right|$

   N.B. Notice that we ignore the signs of the components of the complex number when finding the initial angle.

2. Sketch a quick Argand diagram to decide which quadrant z lies in and then decide what you need to do to the angle found in step 1.

For example, if z is in the 4th quadrant, subtract the angle found from 2π (since $2\pi \equiv 360^{\circ}$).

E.g. 1 Convert $-1-\sqrt{3}\mathrm{i}$ to $[r,\ \theta]$ form.

Working: $r = \left|-1-\sqrt{3}\mathrm{i}\right| = \sqrt{(-1)^2+(-3)^2} = 2$

Ignore the signs: $\tan^{-1}\dfrac{\sqrt{3}}{1} = \dfrac{\pi}{3}$

$-1-\sqrt{3}\mathrm{i}$ is in the 3rd quadrant so we need to add π (180°) to the acute angle

$$\text{Arg}\,(-1-\sqrt{3}\mathrm{i}) = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$$

$$-1-\sqrt{3}\mathrm{i} \equiv \left[2, \frac{4\pi}{3}\right]$$

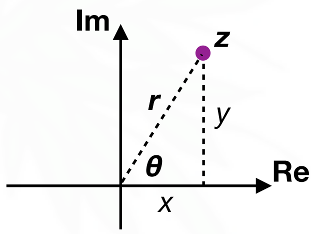
Modulus-argument to Cartesian form

E.g. 2 For the complex number $z = [r,\ \theta]$, express the $x-$ and $y-$ coordinates in terms of r and θ. Use the diagram to help you.

This gives us the more common way to express a complex number in modulus-argument form:

$$z = r\cos\theta + \mathrm{i}\,r\sin\theta = r(\cos\theta + \mathrm{i}\sin\theta)$$

This is shortened to $z = r\,\text{cis}\,\theta$.



E.g. 3 Express the complex number $4\ \text{cis}\ \dfrac{\pi}{6}$ in Cartesian form.

Video: Modulus-argument form of complex number

Solutions to Starter and E.g.s

Exercise
p127 4E Qu 1i, 2i, 3i, 4i, 5i, 6i, 7-11

Summary

Cartesian form $x + y\mathrm{i}$

$x = r\cos\theta$ and $y = r\sin\theta$

Modulus-argument form $r\ \text{cis}\ \theta$ where $r \geq 0$ and $0 \leq \theta < 2\pi$ **or** $-\pi < \theta \leq \pi$

Modulus $r = \sqrt{x^2+y^2}$

Argument 1st quadrant: $\theta = \tan^{-1}\left|\dfrac{y}{x}\right|$

2nd quadrant: $\theta = \pi - \tan^{-1}\left|\dfrac{y}{x}\right|$

3rd quadrant: $$\theta = \pi + \tan^{-1}\left|\frac{y}{x}\right|$$

4th quadrant: $$\theta = 2\pi - \tan^{-1}\left|\frac{y}{x}\right|$$