Operations in Modulus-Argument Form

Starter

1. **(Review of last lesson)** Use an Argand diagram to find, in the form $a + b\text{i}$, the complex number(s) which satisfies arg $(z - 4\text{i}) = \pi$ and $|z + 6| = 5$.

2. Let $z_1 = r_1(\cos\theta_1 + \text{i}\ \sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + \text{i}\ \sin\theta_2)$
State the values of:
(a) $|z_1|$ (b) $|z_2|$ (c) $|z_1 \times z_2|$ (d) $\left|\dfrac{z_1}{z_2}\right|$

Notes

To find how we find the argument is affected we need two trigonometric identities that you meet in the AS and A2 maths course, namely:

AS: $\cos^2\theta + \sin^2\theta = 1$
A2: $\sin(A + B) = \sin A\cos B + \cos A\sin B$
A2: $\cos(A + B) = \cos A\cos B - \sin A\sin B$
A2: $\sin(A - B) = \sin A\cos B - \cos A\sin B$
A2: $\cos(A - B) = \cos A\cos B + \sin A\sin B$

E.g. 1 Let $z_1 = r_1(\cos\theta_1 + \text{i}\ \sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + \text{i}\ \sin\theta_2)$.
(a) State the values of: (i) arg z_1 (ii) arg z_2
(b) Find the values of: (i) arg $(z_1 \times z_2)$ (ii) arg $\left(\dfrac{z_1}{z_2}\right)$

Working: (a) (i) θ_1
(ii) θ_2

(b) (i) $z_1 \times z_2 = r_1(\cos\theta_1 + \text{i}\ \sin\theta_1) \times r_2(\cos\theta_2 + \text{i}\ \sin\theta_2)$
$= r_1r_2\big(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + \text{i}\ (\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\big)$
$= r_1r_2\big(\cos(\theta_1 + \theta_2) + \text{i}\ (\sin(\theta_1 + \theta_2)\big)$
So arg $(z_1 \times z_2) = \theta_1 + \theta_2$

(ii) To find the arg $\left(\dfrac{z_1}{z_2}\right)$ we must multiply the numerator and denominator by the complex conjugate of the denominator.

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + \text{i}\ \sin\theta_1)}{r_2(\cos\theta_2 + \text{i}\ \sin\theta_2)} \times \frac{r_2(\cos\theta_2 - \text{i}\ \sin\theta_2)}{r_2(\cos\theta_2 - \text{i}\ \sin\theta_2)}$$
$$= \frac{r_1}{r_2}\frac{\big(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 + \text{i}\ (\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)\big)}{\cos^2\theta_2 + \sin^2\theta_2}$$
$$= \frac{r_1}{r_2}\big(\cos(\theta_1 - \theta_2) + \text{i}\ (\sin(\theta_1 - \theta_2)\big)$$

So arg $\left(\dfrac{z_1}{z_2}\right) = \theta_1 - \theta_2$

Summary

- $|z_1 \times z_2| = |z_1| \times |z_2|$
- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
- $\arg(z_1 \times z_2) = \theta_1 + \theta_2$
- $\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2$

N.B. When an argument is outside $0 \le \theta < 2\pi$, add or subtract multiples of 2π until the angle falls within the required range.

E.g. 2 Let $z_1 = -1 + \sqrt{3}\mathrm{i}$ and $z_2 = \sqrt{3} + \mathrm{i}$. By giving your answers $0 \le \theta < 2\pi$, find:

(a) $|z_1 \times z_2|$ and Arg $(z_1 \times z_2)$

(b) $|z_1 \div z_2|$ and Arg $(z_1 \div z_2)$

(c) Arg (z_1^4)

Working: (c) $\text{Arg}(z_1^4) = 4 \times \text{Arg}\, z_1 = 4 \times \frac{2\pi}{3} = \frac{8\pi}{3}$

$\frac{8\pi}{3}$ is outside the range $0 \le \theta < 2\pi$ so subtract 2π

$\frac{8\pi}{3} - 2\pi = \frac{2\pi}{3}$

$\text{Arg}(z_1^4) = \frac{2\pi}{3}$

Video: Operations in modulus-argument form

Complex numbers EQ

Solutions to Starter and E.g.s

Exercise

p137 4G Qu 1i, 2i, 3-5 (6 requires trigonometric identities)

Summary

- $|z_1 \times z_2| = |z_1| \times |z_2|$
- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
- $\arg(z_1 \times z_2) = \theta_1 + \theta_2$
- $\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2$

N.B. When an argument is outside $0 \le \theta < 2\pi$, add or subtract multiples of 2π until the angle falls within the required range.