Scalar product and angles

Starter

1. Find the position vector of the point of intersection of $\dfrac{x+2}{4} = y = \dfrac{z-1}{2}$ and $\dfrac{x}{2} = \dfrac{y+1}{2} = 4 - z$.

2. Find the magnitude (modulus) of the following vectors:

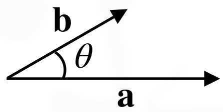
(a) $\begin{pmatrix} -6 \\ 5 \end{pmatrix}$ (b) $5\mathbf{i} - 3\mathbf{j} + \mathbf{k}$

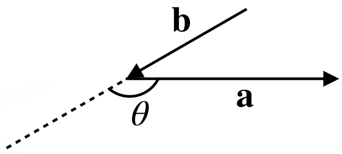
Notes

The scalar product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is defined as:

$\mathbf{a} \cdot \mathbf{b} = |a||b|\cos\theta$ where θ is the angle between the vectors

Notice that the vectors must be pointed away from where they would intersect.





N.B. The result of the scalar product of two vectors is a scalar i.e. a number.

Finding the scalar product of two vectors when we do not know the angle between them

E.g. 1 (a) Using the definition of the scalar product, $\mathbf{a} \cdot \mathbf{b} = |a||b|\cos\theta$, write down the values of: (i) $\mathbf{i} \cdot \mathbf{i}$ (ii) $\mathbf{i} \cdot \mathbf{j}$

(b) Hence write down the values of:
(i) $\mathbf{j} \cdot \mathbf{j}$ (ii) $\mathbf{k} \cdot \mathbf{k}$ (iii) $\mathbf{i} \cdot \mathbf{k}$ (iv) $\mathbf{j} \cdot \mathbf{k}$

(c) Find the value of $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$.

Component form of scalar product: $\mathbf{u} \cdot \mathbf{v} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = u_1v_1 + u_2v_2 + u_3v_3$

E.g. 2 Find the scalar products of the following pairs of vectors:

(a) $\begin{pmatrix} 4 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 10 \end{pmatrix}$ (b) $(3\mathbf{i} + 5\mathbf{j}) \cdot (8\mathbf{i} - 5\mathbf{j})$

Working: (a) $\begin{pmatrix} 4 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 10 \end{pmatrix} = 3 \times 4 + 6 \times 10 = 72$

E.g. 3 Given that $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, find the value of $\mathbf{v} \, . \, \mathbf{v}$ in terms of $\mathbf{v}$?

Working: $\mathbf{v} \, . \, \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} . \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = v_1^2 + v_2^2 + v_3^2$

But $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$

So $\mathbf{v} \, . \, \mathbf{v} = |\mathbf{v}|^2$

Calculating the angle between two vectors

E.g. 4 Using the scalar product find the angle between the two vectors $\begin{pmatrix} 5 \\ 7 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ -2 \end{pmatrix}$.

Working: $\begin{pmatrix} 5 \\ 7 \end{pmatrix} . \begin{pmatrix} 6 \\ -2 \end{pmatrix} = 5 \times 6 + 7 \times (-2) = 16$

$\left| \begin{pmatrix} 5 \\ 7 \end{pmatrix} \right| = \sqrt{5^2 + 7^2} = \sqrt{74}$ and $\left| \begin{pmatrix} 6 \\ -2 \end{pmatrix} \right| = \sqrt{6^2 + (-2)^2} = 2\sqrt{10}$

Using $\cos\theta = \dfrac{\mathbf{a} \, . \, \mathbf{b}}{|a||b|}$: $\cos\theta = \dfrac{16}{\sqrt{74} \times 2\sqrt{10}}$

The angle between the vectors is $72.9°$.

The angle between two vectors can be found using the scalar product: $\cos\theta = \dfrac{\mathbf{a} \, . \, \mathbf{b}}{|a||b|}$

E.g. 5 For each of the following pairs of vectors find the angle between them:

(a) $3\mathbf{i} - \mathbf{j}$ and $5\mathbf{i} - 2\mathbf{j}$ (b) $\begin{pmatrix} -2 \\ 8 \end{pmatrix}$ and $\begin{pmatrix} -5 \\ -9 \end{pmatrix}$

E.g. 6 What can be said about the angle between the two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ if:

(a) $\mathbf{a} \, . \, \mathbf{b} = 0$ (b) $\mathbf{a} \, . \, \mathbf{b} > 0$ (b) $\mathbf{a} \, . \, \mathbf{b} < 0$

Working: (a) $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors $\Rightarrow$ $|\mathbf{a}| \neq 0$ and $|\mathbf{b}| \neq 0$

So if $\mathbf{a} \, . \, \mathbf{b} = 0$ $\Rightarrow$ $\cos\theta = 0$ $\Rightarrow$ $\theta = 90°$

Therefore, the vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular to one another.

Angle between two lines given in vector form

E.g. 7 Find the acute angle between the lines $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 7 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 5 \end{pmatrix}$

Working: Since each line is parallel to its direction vector, we can find the angle between the lines by finding the angle between the direction vectors.

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} . \begin{pmatrix} -3 \\ 5 \end{pmatrix} = 1 \times (-3) + 2 \times 5 = 7$$

$$\left| \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

$$\left| \begin{pmatrix} -5 \\ -9 \end{pmatrix} \right| = \sqrt{(-3)^2 + 5^2} = \sqrt{34}$$

Using $\cos\theta = \dfrac{\mathbf{a} . \mathbf{b}}{|a||b|}$: $\cos\theta = \dfrac{7}{\sqrt{5} \times \sqrt{34}}$

The angle between the lines is $57.5°$.

To find the ***angle between two lines*** in vector form, find the ***angle between the direction vectors***.

E.g. 8 Find the angles between the following pairs of lines:

(a) $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} - 2\mathbf{k})$ and $\mathbf{r} = \mathbf{i} - 2\mathbf{j} + \mathbf{k} + \mu(7\mathbf{i} + 4\mathbf{j} - 4\mathbf{k})$

(b) $\mathbf{r} = (2 + 6\lambda)\mathbf{i} + (1 - 3\lambda)\mathbf{j} + 2\lambda\mathbf{k}$ and $\mathbf{r} = (2 + 4\mu)\mathbf{i} + 7\mathbf{j} + (5 + 3\mu)\mathbf{k}$

Video: Scalar product
Video: Angle between 2 lines

Scalar product EQ

Solutions to Starter and E.g.s

Exercise

p54 2D Qu 1i, 2i, 3i, 4a, 5i, 6i, 7-11

Summary

Definition of scalar product: $\mathbf{a} . \mathbf{b} = |a||b|\cos\theta$

Component form of scalar product: $\mathbf{u} . \mathbf{v} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} . \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = u_1v_1 + u_2v_2 + u_3v_3$

The angle between two vectors can be found using the scalar product: $\cos\theta = \dfrac{\mathbf{a} . \mathbf{b}}{|a||b|}$

If $\mathbf{a} . \mathbf{b} = 0$ $\Rightarrow$ the vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular to one another.
If $\mathbf{a} . \mathbf{b} > 0$ $\Rightarrow$ the angle between the vectors is acute.
If $\mathbf{a} . \mathbf{b} < 0$ $\Rightarrow$ the angle between the vectors is obtuse.
To find the angle between two lines in vector form, find the angle between the direction vectors.