Transformations in 3-D

Starter

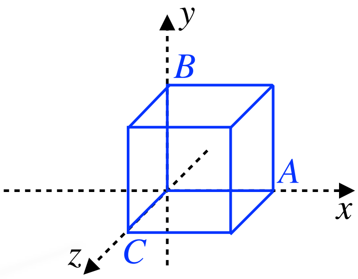
1. The matrix $\mathbf{M} = \begin{pmatrix} 2 & -1 \\ 2 & 4 \end{pmatrix}$ represents a transformation.

   (a) Find all the invariant points of $\mathbf{M}$.

   (b) Show that $\mathbf{M}$ has no invariant lines through the origin.

2. Prove that the line $y = 2x - 1$ is mapped onto itself under the transformation $\begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}$.

3. By considering the unit cube, find the matrix which causes a reflection in the $y - z$ plane.
   Hint: The $y - z$ plane is also called the $x = 0$ plane.



Notes

Rotations and reflections in $2-$D and $3-$D are different in the following ways:

	Shapes rotate about a…	Shapes are reflected in a…	Consider the…
2-D	…point	…line	…unit square
3-D	…line	…plane	…unit cube

Reflections

E.g. 1 From the starter, the matrix which causes a reflection in the $y - z$ plane (or $x = 0$ plane) is $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Hence state the matrix that causes a reflection in:

(a) $x - z$ plane or $y = 0$ plane (b) $x - y$ plane or $z = 0$ plane

Working: (a) Look at where the -1 goes.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

These do not appear in the formula booklet.

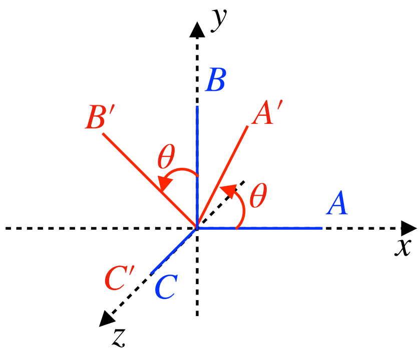
Rotations

E.g. 2 Find the matrix $\mathbf{M}$ such that the point $P(x, y, z)$ is rotated $90°$ about the $z-$axis.

E.g. 3 (a) Find the matrix $\mathbf{M}$ such that the point $P(x, y, z)$ is rotated $\theta°$ about the $z-$axis.
Hint: A positive $\theta°$ means an anti-clockwise rotation.

(b) Hence write down matrices that cause rotations of $\theta°$ about the:
(i) $x-$axis (ii) $y-$axis

Working: (a)



N.B. These do appear in the formula booklet.

Importance of the determinant
The determinant gives the volume factor for the enlargement.
If the determinant is negative, the transformation changes the orientation of the object.

E.g. 4 Find the effect on volume and orientation of the transformations defined by these matrices:

(a) $\begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & 4 \\ 0 & -2 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} -1 & 3 & -1 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{pmatrix}$

Working: (a) $\left|\begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & 4 \\ 0 & -2 & 0 \end{pmatrix}\right| = \left|\begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & 4 \\ 0 & -2 & 0 \end{pmatrix}^T\right| = \left|\begin{pmatrix} 2 & 0 & 0 \\ -1 & 1 & -2 \\ 1 & 4 & 0 \end{pmatrix}\right|$

$= 16$

The transformation enlarges the volume by a factor of 16 but does not change an objects' orientation.

Enlargements and stretches
E.g. 5 Find the matrix in 3-D which causes:
(a) an enlargement by a factor 2,
(b) a stretch by a factor of 3 in the $x-$and $z-$directions.

Working: (a) $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$

So the matrix is $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

Compound transformations
E.g. 6 Find the matrix which causes an enlargement scale factor 4 followed by a reflection in the $y - z$ plane.

Video: Matrix transformations in 3-D
Solutions to Starter and E.g.s

Exercise
p92 3E Qu 1ac, 2ac, 3-11, 13

Summary

Reflection (in the given plane)		
$x=0$ or $y-z$ plane	$y=0$ or $x-z$ plane	$z=0$ or $x-y$ plane
$\begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}$

Rotations (about the given axis)		
$x-$axis	$y-$axis	$z-$axis
$\begin{pmatrix}1&0&0\\0&\cos\theta&-\sin\theta\\0&\sin\theta&\cos\theta\end{pmatrix}$	$\begin{pmatrix}\cos\theta&0&\sin\theta\\0&1&0\\-\sin\theta&0&\cos\theta\end{pmatrix}$	$\begin{pmatrix}\cos\theta&-\sin\theta&0\\\sin\theta&\cos\theta&0\\0&0&1\end{pmatrix}$