Transforming equations

Starter

1. **(Review of last lesson)**
The roots of the equation $ax^2 + bx + c = 0$ differ by 1. Prove that $b^2 - a^2 - 4ac = 0$.

Notes

Often the following method is better than the "roots method" when questions are harder.

Substitution method

The substitution method will be shown with a previous example.

E.g. The quadratic equation $x^2 + 5x + 7 = 0$ has roots α and β. Without calculating α and β, find an equation with roots 2α and 2β.

Working: Let u be one of the new roots so $u = 2\alpha \quad \Rightarrow \quad \alpha = \frac{1}{2}u$.

Since α satisfies $x^2 + 5x + 7 = 0$ then $\left(\frac{1}{2}u\right)^2 + 5\left(\frac{1}{2}u\right) + 7 = 0$

Simplifying: $\frac{1}{4}u^2 + \frac{5}{2}u + 7 = 0$

$\therefore$ new equation is $u^2 + 10u + 28 = 0$

Since this is true if u is either 2α or 2β, $u^2 + 10u + 28 = 0$ is the required equation.

N.B. Equations $u^2 + 10u + 28 = 0$ and $x^2 + 10x + 28 = 0$ both have the required roots so either answer would be acceptable.

E.g. 1 The equation $x^2 + 4x + 7 = 0$ has roots α and β. Use the "substitution method" to find equations with integer coefficients which have the following roots.

(a) 3α and 3β (b) α^2 and β^2 (c) $\alpha + 2\beta$ and $2\alpha + \beta$

Working:

(c) Let $u = \alpha + 2\beta$ so $\alpha = u - 2\beta$

$(u - 2\beta)^2 + 4(u - 2\beta) + 7 = 0$

Expand: $u^2 + 4(1 - \beta)u + 4\beta(\beta - 2) + 7 = 0$

What is β? This method doesn't seem to work so let's try…

If $u = \alpha + 2\beta$ then:

$u + \alpha = 2\alpha + 2\beta = 2(\alpha + \beta) = 2 \times -4 = -8$

Remember: $\alpha + \beta = -\frac{b}{a} = -4$

So $\alpha = -u - 8$

Substitute: $(-u - 8)^2 + 4(-u - 8) + 7 = 0$

$u^2 + 12u + 39 = 0$

The above methods also work for cubics and quartics.

E.g. 2 The equation $2x^3 + 3x^2 + 4x + 5 = 0$ has roots α, β and γ. Find the equation whose roots are α^{-1}, β^{-1} and γ^{-1}.

Working: Let u be one of the new roots so $u = \alpha^{-1} \quad \Rightarrow \quad \alpha = u^{-1}$
$2(u^{-1})^3 + 3(u^{-1})^2 + 4u^{-1} + 5 = 0$
Multiplying by u^3 gives: $5u^3 + 4u^2 + 3u + 2 = 0$

E.g. 3 The equation $x^3 + 3x^2 + 4x + 5 = 0$ has roots α, β and γ. Find the equation whose roots are $\beta + \gamma$, $\gamma + \alpha$ and $\alpha + \beta$.
Hint: Similar to E.g. 1(c)

E.g. 4 The equation $x^3 - 9x^2 + 31x - 39 = 0$ has roots α, β and γ which are in arithmetic progression. Solve the equation.
N.B. From GCSE, arithmetic progression means that terms are found by adding the same amount each time e.g. $a,\ a + d,\ a + 2d, \ldots.$
Hint: Rather than α, $\alpha + d$ and $\alpha + 2d$ let the roots be $\alpha - d$, α and $\alpha + d$

E.g. 5 The roots of the equation $x^3 + ax^2 + bx + c = 0$, where $c \neq 0$ are in geometric progression. Prove that $b^3 = a^3c$.
N.B. From GCSE, geometric progression means that terms are found by multiplying by the same number each time e.g. $a,\ ar,\ ar^2, \ldots.$
Hint: Rather than α, αr and αr^2 let the roots be $\frac{\alpha}{r}$, α and αr

Video: Transforming equations

Solutions to Starter and E.g.s

Exercise
p162 5F Qu 1i, 2-8, (9-12 red)