Vector equation of a line

Starter

1. A 3-D transformation is defined as a stretch by a factor of 2 in the $x-$direction, followed by a rotation of $45°$ about the $z-$axis. Find the matrix which defined this mapping.

Notes

Position vectors

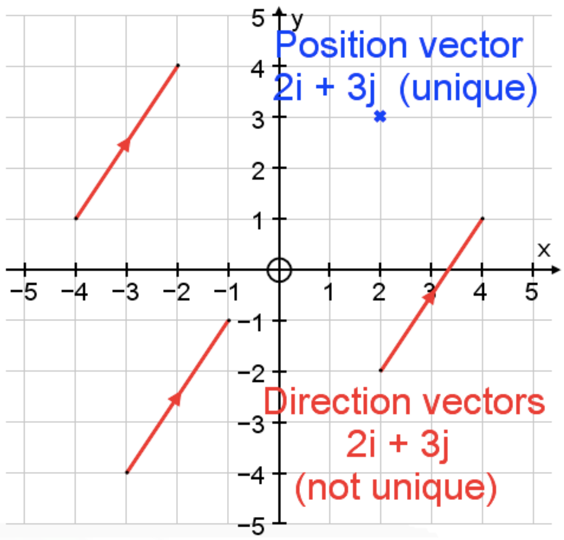
A position vector is a unique point in space with respect to the origin.

So the point P(2, 3) has position vector $\overrightarrow{OP} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

Displacement (or direction) vectors

A displacement vector is a vector that connects two points.

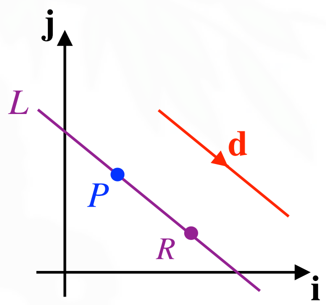
The displacement vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is not unique and can be placed anywhere in space. Obviously all the displacement vectors $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ are parallel.



The distinction between position and displacement vectors is not hugely important but it does allow the addition of points and vectors which is the key idea.

Equation of a line

Let $\mathbf{p}$ be the position vector of any fixed point, P, on the line, L, i.e. such that $OP = \mathbf{p}$.
Let $\mathbf{d}$ be a direction vector ***parallel to the line***.
Let any point, R, on the line have position vector $\mathbf{r}$.
The vector $\overrightarrow{PR}$ lies in the line so $\overrightarrow{PR}$ is parallel to $\mathbf{d}$.



i.e. $\overrightarrow{PR}$ is a multiple of $\mathbf{d}$ $\quad \Rightarrow \quad \overrightarrow{PR} = \lambda\mathbf{d}$ $\quad$ so $\mathbf{r} - \mathbf{p} = \lambda\mathbf{d}$

$\mathbf{r} = \mathbf{p} + \lambda\mathbf{d}$ where $\mathbf{p}$ is the position vector of any fixed point on the line,
λ is a scalar ($\lambda \in \mathbb{R}$)
and $\mathbf{d}$ is any direction vector parallel to the line

What is λ? By giving λ different values we generate the points that make up the line. Each value of λ gives a unique point on the line.

E.g. 1 Find the vector equation of the line passing through $A(3, 2, 1)$ and $B(1, 4, 2)$.

Working: The equation is of the form $\mathbf{r} = \mathbf{p} + \lambda \mathbf{d}$

To find $\mathbf{d}$ we can find $\overrightarrow{AB}$ or $\overrightarrow{BA}$ and $\mathbf{p}$ can be either point

Either $\mathbf{p} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ or $\mathbf{p} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$

Either $\mathbf{d} = \overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$

…or… $\mathbf{d} = \overrightarrow{BA} = \mathbf{a} - \mathbf{b} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$

The equation of the line could be $\mathbf{r} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$.

N.B. Other combinations of the vectors above are also possible.

N.B. Since multiples of $\mathbf{d}$ are possible, a line can be expressed in an infinite number of ways in vector form. This is similar to the Cartesian form i.e. $x + y = 4 \equiv 2x + 2y = 8$ etc. Multiples of the direction vector are fine but multiples of the position vector would (generally) be incorrect.

E.g. 2 Find the equation of the line that passes through the points $(7, 2)$ and $(-5, 6)$.

Showing a point lies on the line

E.g. 2 Decide whether the point $P(9, 5, 8)$ lies on the line $\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}$.

Success criteria — deciding whether a point lies on a line

1. Put the point equal to the line and form linear equations for each component ($\mathbf{i}$, $\mathbf{j}$ & $\mathbf{k}$).
2. Solve each of the linear equations.
   λ is the same for all equations $\Rightarrow$ the point lies on the line.
   λ is ***not*** the same for all equations $\Rightarrow$ the point does ***not*** lie on the line.

E.g. 3 Decide whether $\mathbf{r} = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and $\mathbf{r} = \begin{vmatrix} 9 \\ 0 \\ 8 \end{vmatrix} + \mu \begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix}$ are the same line or not.

Working: $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ is a multiple of $\begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix}$ so the lines are at least parallel.

Does the point $\begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}$ lie on the line $\mathbf{r} = \begin{vmatrix} 9 \\ 0 \\ 8 \end{vmatrix} + \mu \begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix}$?

$$\begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 \\ 0 \\ 8 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix}$$

Equating components:

$$\mathbf{i}: \quad 5 = 9 - 2\mu \quad \Rightarrow \quad \mu = 2$$
$$\mathbf{j}: \quad 2 = \mu$$
$$\mathbf{k}: \quad 1 = 8 - 3\mu \quad \Rightarrow \quad \mu = \frac{7}{3}$$

Since the $\mu-$values are not all the same, the fixed point of one line does not lie on the other line. Therefore, they are not the same line.

Success criteria — deciding whether two equations of lines are equivalent

1. Compare direction vectors — if they are multiples of each other, proceed to step 2. If they are not multiples, the lines are not even parallel and so are not the same line.
2. Check whether the fixed point of one line lies on the other line — so put the fixed point of one line equal to the equation of the other line and solve the equations for λ. If the values are all equal, the lines are equivalent.

E.g. 4 Decide whether $\mathbf{r} = \begin{pmatrix} -3 \\ -3 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} -6 \\ -8 \\ 4 \end{pmatrix}$ are the same line or not.

Video: Vector equation of lines
Video: Parallel lines

Solutions to Starter and E.g.s

Exercise
p38 2A Qu 1i, 2i, 3i, 4i, 5-11

Summary
The vector equation of the line is of the form:

$\mathbf{r} = \mathbf{p} + \lambda \mathbf{d}$ where $\mathbf{p}$ is the position vector of any fixed point on the line,
λ is a scalar ($\lambda \in \mathbb{R}$)
and $\mathbf{d}$ is any direction vector parallel to the line

Success criteria — deciding whether a point lies on a line

1. Put the point equal to the line and form linear equations for each component ($\mathbf{i}$, $\mathbf{j}$ & $\mathbf{k}$).
2. Solve each of the linear equations.
   λ is the same for all equations $\Rightarrow$ the point lies on the line.
   λ is ***not*** the same for all equations $\Rightarrow$ the point does ***not*** lie on the line.

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