The vector product

Starter

1. Find the angle between lines $\mathbf{r} = (5 - 2\lambda)\mathbf{i} + 4\lambda\mathbf{j}$ and $\mathbf{r} = (5 + \mu)\mathbf{i} + (3 - 7\mu)\mathbf{j}$.

2. If $\mathbf{a}$ and $\mathbf{b}$ are perpendicular, simplify:
   (a) $(\mathbf{a} - \mathbf{b}) . (\mathbf{a} - \mathbf{b})$
   (b) $(\mathbf{a} - \mathbf{b}) . \mathbf{b} - (\mathbf{b} - \mathbf{a}) . \mathbf{a}$
   (c) $(2\mathbf{a} + 3\mathbf{b}) . \mathbf{b}$

3. The vector $\begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix}$ is perpendicular to both $\begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 5 \\ 6 \end{pmatrix}$. Find α and β.

Notes

To find a vector perpendicular to two vectors in 3-dimensions, it is possible to use simultaneous questions like in question 3 from the starter. However, it is faster to use the ***vector product***.

Definition of vector product

The definition of the vector product for two vectors $\mathbf{a}$ and $\mathbf{b}$ is:

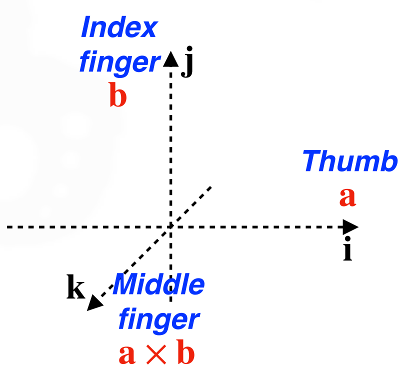
$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}}$ where $\hat{\mathbf{n}}$ is the ***unit vector perpendicular*** to both $\mathbf{a}$ and $\mathbf{b}$.

N.B. The result when combining two vectors using the vector product is a vector.
Since the resultant vector, $\mathbf{a} \times \mathbf{b}$, is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, the vector product does not work in 2—dimensions.

Right-handed rule

The right-handed rule helps to decide the direction of the resultant vector of $\mathbf{a} \times \mathbf{b}$.
Using your right hand, hold your thumb, index finger and middle finger at right-angles to each other.

For coordinate axes in $3-$dimensions:
- the positive direction of the $\mathbf{i}-$axis is the thumb
- the positive direction of the $\mathbf{j}-$axis is the index finger
- the positive direction of the $\mathbf{k}-$axis is the middle finger



With the $\mathbf{i}-$ and $\mathbf{j}-$axes drawn normally on a whiteboard, the $\mathbf{k}-$axis is defined as coming out of the whiteboard.

With the vector product $\mathbf{a} \times \mathbf{b}$:
- the direction of the vector $\mathbf{a}$ is the thumb
- the direction of the vector $\mathbf{b}$ is the index finger
- the direction of the resultant vector, $\mathbf{a} \times \mathbf{b}$, is the middle finger

E.g. 1 Using the definition of the vector product and the right-hand rule state the values of:

(a) (i) $\mathbf{i} \times \mathbf{i}$ (ii) $\mathbf{j} \times \mathbf{j}$ (iii) $\mathbf{k} \times \mathbf{k}$
(b) (i) $\mathbf{i} \times \mathbf{j}$ (ii) $\mathbf{j} \times \mathbf{i}$ (iii) $\mathbf{i} \times \mathbf{k}$
(iv) $\mathbf{k} \times \mathbf{i}$ (v) $\mathbf{j} \times \mathbf{k}$ (vi) $\mathbf{k} \times \mathbf{j}$

Remember: $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

Working: (a) (i) $\mathbf{i} \times \mathbf{i} = |\mathbf{i}||\mathbf{i}|\sin 0 \times \hat{\mathbf{n}} = 0$

(b) (i) $\mathbf{i} \times \mathbf{j} = |\mathbf{i}||\mathbf{k}|\sin 90 \times \hat{\mathbf{n}} = 1 \times 1 \times 1 \times \hat{\mathbf{n}} = \hat{\mathbf{n}}$

Using the right-hand rule where $\mathbf{i}$ is the thumb and $\mathbf{j}$ is the index finger, it means that the resultant vector is in the direction of the middle finger i.e. the $\mathbf{k}-$direction.

$\mathbf{i} \times \mathbf{j} = \mathbf{k}$

To summarise:

$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$

$\mathbf{i} \times \mathbf{j} = \mathbf{k}$ ***but*** $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$

$\mathbf{i} \times \mathbf{k} = -\mathbf{j}$ ***but*** $\mathbf{k} \times \mathbf{i} = \mathbf{j}$

$\mathbf{j} \times \mathbf{k} = \mathbf{i}$ ***but*** $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$

E.g. 2 Find $(3\mathbf{i} + 2\mathbf{k}) \times (-4\mathbf{i} + 5\mathbf{j})$.

Working:

$$(3\mathbf{i} + 2\mathbf{k}) \times (-4\mathbf{i} + 5\mathbf{j}) = -12\mathbf{i} \times \mathbf{i} + 15\mathbf{i} \times \mathbf{j} - 8\mathbf{k} \times \mathbf{i} + 10\mathbf{k} \times \mathbf{j}$$
$$= -10\mathbf{i} - 8\mathbf{j} + 15\mathbf{k}$$

To make the process quicker we find the vector product of two vectors using the determinant of a 3 by 3 matrix method.

For the vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$: $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

E.g. 3 Find $\mathbf{p} \times \mathbf{q}$ when $\mathbf{p} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{q} = 4\mathbf{i} - \mathbf{j} + 7\mathbf{k}$

Working:

$$\mathbf{p} \times \mathbf{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 4 & -1 & 7 \end{vmatrix} = 17\mathbf{i} + 5\mathbf{j} - 9\mathbf{k}$$

E.g. 4 Find $\mathbf{a} \times \mathbf{b}$ when $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 5\mathbf{i} + 4\mathbf{j} - \mathbf{k}$.

E.g. 5 Write down the relationship between $\mathbf{a} \times \mathbf{b}$ and $\mathbf{b} \times \mathbf{a}$?

E.g. 6 What does $\mathbf{a} \,.\, (\mathbf{a} \times \mathbf{b})$ equal? Hence write down the value of $\mathbf{b} \,.\, (\mathbf{a} \times \mathbf{b})$.

E.g. 7 Find a unit vector which is perpendicular to both $\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

E.g. 8 Given that $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$, $\mathbf{b} = p\mathbf{j} + q\mathbf{k}$ and $\mathbf{a} \times \mathbf{b} = 2\mathbf{j} + \lambda\mathbf{k}$, find the values of the scalars p, q and λ.

Additional results

- $\sin\theta = \dfrac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}||\mathbf{b}|}$
- Distributive rule: $(\mathbf{p}+\mathbf{q})\times\mathbf{r} = \mathbf{p}\times\mathbf{r}+\mathbf{q}\times\mathbf{r}$
- $\mathbf{a}\,.\,(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\,.\,(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\,.\,(\mathbf{a}\times\mathbf{b}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

Video: Vector product

Vectors EQ

Solutions to Starter and E.g.s

Exercise
p59 2E Qu 1i, 2i, 4-8

Summary
Definition of vector product: $\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}}$ where $\hat{\mathbf{n}}$ is the ***unit vector perpendicular*** to both $\mathbf{a}$ and $\mathbf{b}$.

Right-handed rule
With the vector product $\mathbf{a}\times\mathbf{b}$:
the direction of the vector $\mathbf{a}$ is the thumb
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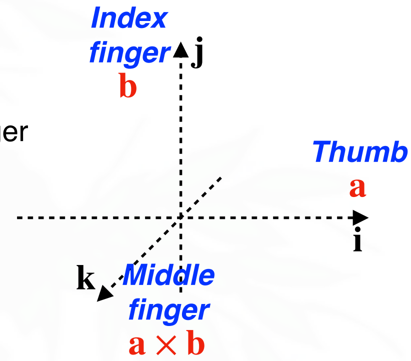

$\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = \mathbf{0}$
$\mathbf{i}\times\mathbf{j} = \mathbf{k}$ *but* $\mathbf{j}\times\mathbf{i} = -\mathbf{k}$
$\mathbf{i}\times\mathbf{k} = -\mathbf{j}$ *but* $\mathbf{k}\times\mathbf{i} = \mathbf{j}$
$\mathbf{j}\times\mathbf{k} = \mathbf{i}$ *but* $\mathbf{k}\times\mathbf{j} = -\mathbf{i}$

For the vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$: $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$\mathbf{a}\times\mathbf{b} = -\,\mathbf{b}\times\mathbf{a}$

$\sin\theta = \dfrac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}||\mathbf{b}|}$