Topic X2 Vectors and induction (Post-TT A) [40]

1. Part (i) only.

Two skew lines have equations

$$\frac{x}{2} = \frac{y+3}{1} = \frac{z-6}{3} \quad \text{and} \quad \frac{x-5}{3} = \frac{y+1}{1} = \frac{z-7}{5}.$$

(i) Find the direction of the common perpendicular to the lines. **[2]**

(ii) Find the shortest distance between the lines. **[4]**

(Total 2 marks)

2.

Given that the vectors **a** and **b** are perpendicular, prove that

$|(\mathbf{a} + 5\mathbf{b}) \times (\mathbf{a} - 4\mathbf{b})| = k|\mathbf{a}||\mathbf{b}|$, where k is an integer to be found.

Explicitly state any properties of the vector product that you use within your proof.

(Total 9 marks)

3.

(i) Prove by induction that, for $n \in \mathbb{N}$,

$$\begin{pmatrix} 4 & -1 \\ 9 & -2 \end{pmatrix}^n = \begin{pmatrix} 3n+1 & -n \\ 9n & 1-3n \end{pmatrix}$$

(6)

(ii) Consider the statement

$$n^2 < 2^n \quad \text{for all } n \in \mathbb{Z}^+$$

A student attempts to prove this statement using induction as follows.

Student's response

For $n = 1$ we have $1^2 = 1$ and $2^1 = 2$
Since $1 < 2$ the statement is true for $n = 1$

Suppose it is true for $n = k$, so $k^2 < 2^k$

Line 4 → Then $(k + 1)^2 = k^2 + 2k + 1 < k^2 + k^2$ (since $2k + 1 < k^2$ for $k \in \mathbb{Z}^+$)

$$= 2k^2$$
$$< 2 \times 2^k \quad \text{(by the assumption } k^2 < 2^k\text{)}$$
$$= 2^{k+1}$$

Hence the result is true for $n = k + 1$

So the result is true for $n = 1$ and if it is true for $n = k$ then it is true for $n = k + 1$, and hence it is true for all positive integers n by mathematical induction.

(a) Show by a counterexample that the statement is not true.

Given that the only mathematical error in the student's proof occurs in line 4,

(b) identify the error made in the student's proof,

(c) hence determine for which positive integers the statement is true, explaining your reasoning.

(5)

(Total 11 marks)

4.

Lines l_1 and l_2 have vector equations

$$\mathbf{r} = -\mathbf{i} + 2\mathbf{j} + 7\mathbf{k} + t(2\mathbf{i} + 2\mathbf{j} + \mathbf{k}) \quad \text{and} \quad \mathbf{r} = 2\mathbf{i} + 9\mathbf{j} - 4\mathbf{k} + s(\mathbf{i} + 3\mathbf{j} - 2\mathbf{k})$$

respectively. The point A has coordinates $(-3, 0, 6)$ relative to the origin O.

(i) Show that A lies on l_1 and that OA is perpendicular to l_1. **[3]**

(ii) Show that the line through O and A intersects l_2. **[4]**

(iii) Given that the point of intersection in part **(ii)** is B, find the ratio $|\overrightarrow{OA}| : |\overrightarrow{BA}|$. **[3]**

(Total 10 marks)

5.

The position vectors of three points A, B and C relative to an origin O are given respectively by

$$\overrightarrow{OA} = 7\mathbf{i} + 3\mathbf{j} - 3\mathbf{k},$$
$$\overrightarrow{OB} = 4\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$$
$$\text{and} \quad \overrightarrow{OC} = 5\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}.$$

(i) Find the angle between AB and AC. [6]

(ii) Find the area of triangle ABC. [2]

(Total 8 marks)