Cartesian equation of a line

Starter

1. Find the vector equation of the line passing through $A(1, 3, 5)$ and $B(2, 4, 6)$.

Working: The equation is of the form $\mathbf{r} = \mathbf{p} + \lambda\mathbf{d}$

To find $\mathbf{d}$ we can find $\overrightarrow{AB}$ or $\overrightarrow{BA}$ and $\mathbf{p}$ can be either point

Either $\mathbf{p} = \begin{pmatrix}1\\3\\5\end{pmatrix}$ or $\mathbf{p} = \begin{pmatrix}2\\4\\6\end{pmatrix}$

Either $\mathbf{d} = \overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix}2\\4\\6\end{pmatrix} - \begin{pmatrix}1\\3\\5\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}$

…or… $\mathbf{d} = \overrightarrow{BA} = \mathbf{a} - \mathbf{b} = \begin{pmatrix}1\\3\\5\end{pmatrix} - \begin{pmatrix}2\\4\\6\end{pmatrix} = \begin{pmatrix}-1\\-1\\-1\end{pmatrix}$

The equation of the line could be $\mathbf{r} = \begin{pmatrix}1\\3\\5\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\1\end{pmatrix}$.

N.B. Other combinations of the vectors above are also possible.

2. How do we convert $\mathbf{r} = \begin{pmatrix}p_1\\p_2\end{pmatrix} + \lambda\begin{pmatrix}d_1\\d_2\end{pmatrix}$ from vector form of the line to Cartesian form?

Hint: Let $\mathbf{r} = \begin{pmatrix}x\\y\end{pmatrix}$ and eliminate λ.

Working: $\mathbf{r} = \begin{pmatrix}p_1\\p_2\end{pmatrix} + \lambda\begin{pmatrix}d_1\\d_2\end{pmatrix}$

$x = p_1 + \lambda d_1 \quad \Rightarrow \quad \lambda = \dfrac{x - p_1}{d_1}$

$y = p_2 + \lambda d_2 \quad \Rightarrow \quad \lambda = \dfrac{y - p_2}{d_2}$

Equating gives: $\lambda = \dfrac{x - p_1}{d_1} = \dfrac{y - p_2}{d_2}$

E.g. 1 Convert the vector equation $\mathbf{r} = \begin{pmatrix}p_1\\p_2\\p_3\end{pmatrix} + \lambda\begin{pmatrix}d_1\\d_2\\d_3\end{pmatrix}$ to its Cartesian form.

Working: $x = p_1 + \lambda d_1 \qquad y = p_2 + \lambda d_2 \qquad z = p_3 + \lambda d_3$

Rearranging to make λ the subject:

$\lambda = \dfrac{x - p_1}{d_1} \qquad \lambda = \dfrac{y - p_2}{d_2} \qquad \lambda = \dfrac{z - p_3}{d_3}$

$\dfrac{x - p_1}{d_1} = \dfrac{y - p_2}{d_2} = \dfrac{z - p_3}{d_3} (= \lambda)$

E.g. 2 Convert the following to Cartesian equations:

(a) $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ -7 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$

(b) $\mathbf{r} = \begin{pmatrix} -9 \\ 5 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} -6 \\ \frac{1}{2} \\ 3 \end{pmatrix}$

(c) $\mathbf{r} = \begin{pmatrix} 7 \\ 3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ -8 \\ 5 \end{pmatrix}$

Working: (a) $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ -7 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$

$x = 2 + 4\lambda \quad \Rightarrow \quad \lambda = \frac{x-2}{4}$

$y = 3 - 2\lambda \quad \Rightarrow \quad \lambda = \frac{y-3}{-2}$

$z = -7 + \lambda \quad \Rightarrow \quad \lambda = z + 7$

The Cartesian equation is $\frac{x-2}{4} = \frac{y-3}{-2} = z + 7 (= \lambda)$

(b) $\mathbf{r} = \begin{pmatrix} -9 \\ 5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -6 \\ \frac{1}{2} \\ 3 \end{pmatrix}$

$x = -9 - 6\lambda \quad \Rightarrow \quad \lambda = \frac{x+9}{-6}$

$y = 5 + \frac{1}{2}\lambda \quad \Rightarrow \quad \lambda = 2(y-5)$

$z = 3\lambda \quad \Rightarrow \quad \lambda = \frac{z}{3}$

The Cartesian equation is $\frac{x+9}{-6} = 2(y-5) = \frac{z}{3} (= \lambda)$

(c) $\mathbf{r} = \begin{pmatrix} 7 \\ 3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ -8 \\ 5 \end{pmatrix}$

$x = 7$

$y = 3 - 8\lambda \quad \Rightarrow \quad \lambda = \frac{y-3}{-8}$

$z = -1 + 5\lambda \quad \Rightarrow \quad \lambda = \frac{z+1}{5}$

The Cartesian equation is $x = 7$ and $\frac{y-3}{-8} = \frac{z+1}{5} (= \lambda)$

E.g. 3 Convert the following to vector equations:

(a) $\dfrac{x-2}{3} = \dfrac{y+4}{2} = \dfrac{z-1}{4}$

(b) $4(x+5) = 1 - y = \dfrac{z}{3}$

(c) $\dfrac{x+3}{-7} = 2z$ and $y = -6$

Working: (a)

$\lambda = \dfrac{x-2}{3} \Rightarrow x = 2 + 3\lambda$

$\lambda = \dfrac{y+4}{2} \Rightarrow y = -4 + 2\lambda$

$\lambda = \dfrac{z-1}{4} \Rightarrow z = 1 + 4\lambda$

The vector equation is $\mathbf{r} = \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$

$4(x+5) = 1 - y = \dfrac{z}{3}$

(b)

$\lambda = 4(x+5) \Rightarrow x = -5 + \dfrac{1}{4}\lambda$

$\lambda = 1 - y \Rightarrow y = 1 - \lambda$

$\lambda = \dfrac{z}{3} \Rightarrow z = 3\lambda$

The vector equation is $\mathbf{r} = \begin{pmatrix} -5 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} \frac{1}{4} \\ -1 \\ 3 \end{pmatrix}$ or

$\mathbf{r} = \begin{pmatrix} -5 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -4 \\ 12 \end{pmatrix}$.

(c)

$\lambda = \dfrac{x+3}{-7} \Rightarrow x = -3 - 7\lambda$

$y = -6$

$\lambda = 2z \Rightarrow z = \dfrac{1}{2}\lambda$

The vector equation is $\mathbf{r} = \begin{pmatrix} -3 \\ -6 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -7 \\ 0 \\ \frac{1}{2} \end{pmatrix}$ or

$\mathbf{r} = \begin{pmatrix} -3 \\ -6 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -14 \\ 0 \\ 1 \end{pmatrix}$.

E.g. 4 Let $\mathbf{v} = 5\mathbf{i} + 12\mathbf{j}$. Find the unit vector in the direction of $\mathbf{v}$.

Working: $|\mathbf{v}| = |5\mathbf{i} + 12\mathbf{j}| = \sqrt{5^2 + 12^2} = 13$

So $\hat{\mathbf{v}} = \dfrac{5}{13}\mathbf{i} + \dfrac{12}{13}\mathbf{j}$

E.g. 5 Let $\mathbf{v} = 7\mathbf{i} - 24\mathbf{j}$. Find:

(a) the unit vector and

(b) the vector of length 75 units in the direction of $\mathbf{v}$.

Working: (a) $\hat{\mathbf{v}} = \dfrac{7\mathbf{i} - 24\mathbf{j}}{\sqrt{7^2 + (-24)^2}} = \dfrac{1}{25}(7\mathbf{i} - 24\mathbf{j})$

(b) Vector of length 75 units $= 75 \times \dfrac{1}{25}(7\mathbf{i} - 24\mathbf{j})$

$= 3(7\mathbf{i} - 24\mathbf{j})$

$= 21\mathbf{i} - 72\mathbf{j}$

Video A: Cartesian form of lines

Video B: Cartesian form of lines

Solutions to Starter and E.g.s

Exercise

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