Cubic and Quartic equations

Starter

1. **(Review of last lesson)** Write down a quadratic equation with integer coefficients such that the sum of the roots is $\frac{1}{3}$ and the product is $\frac{1}{2}$.

Working: The simplest equation is $x^2 - \frac{1}{3}x + \frac{1}{2} = 0$.
To get integer coefficients multiply by 6: $6x^2 - 2x + 3 = 0$

2. Let the roots of the cubic equation $ax^3 + bx^2 + cx + d = 0$ be alpha (α), beta (β) and gamma (γ). Therefore, $ax^3 + bx^2 + cx + d \equiv a(x-\alpha)(x-\beta)(x-\gamma)$.
By expanding and equating coefficients, express the following in terms of a, b, c and d.
(a) $\alpha + \beta + \gamma$ (b) $\alpha\beta + \beta\gamma + \gamma\alpha$ (c) $\alpha\beta\gamma$

Working: (a)

$$
\begin{aligned}
ax^3 + bx^2 + cx + d &\equiv a(x-\alpha)(x-\beta)(x-\gamma) \\
&= a(x-\alpha)\left(x^2 - (\beta+\gamma)x + \beta\gamma\right) \\
&= a\left(x^3 - (\beta+\gamma)x^2 + \beta\gamma x - \alpha x^2 + \alpha(\beta+\gamma)x - \alpha\beta\gamma\right) \\
&= a\left(x^3 - (\alpha+\beta+\gamma)x^2 + (\alpha\beta + +\beta\gamma + \gamma\alpha)x - \alpha\beta\gamma\right)
\end{aligned}
$$

Equating coefficients of x^2: $b = -a(\alpha+\beta+\gamma)$

$\therefore \alpha + \beta + \gamma = -\frac{b}{a}$

(b) Equating coefficients of x: $c = a(\alpha\beta + \beta\gamma + \gamma\alpha)$

$\therefore \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$

(c) Equating constant terms: $d = -a(\alpha\beta\gamma)$

$\alpha\beta\gamma = -\frac{d}{a}$

E.g. 1 Find the simplest cubic with roots 2, 3, and 4.

Working: The simplest cubic has 1 as the coefficient of x^3.
$\alpha + \beta + \gamma = 9$ ***(change the sign)***
$\alpha\beta + \beta\gamma + \gamma\alpha = 6 + 12 + 8 = 26$ ***(don't change the sign)***
$\alpha\beta\gamma = 24$ ***(change the sign)***
So the cubic is $x^3 - 9x^2 + 26x - 24 = 0$

E.g. 2 Expand $(\alpha + \beta + \gamma)^2$.

Working:

$$
\begin{aligned}
(\alpha+\beta+\gamma)^2 &= (\alpha+\beta+\gamma)(\alpha+\beta+\gamma) \\
&= \alpha^2 + \alpha\beta + \alpha\gamma + \beta\alpha + \beta^2 + \beta\gamma + \gamma\alpha + \gamma\beta + \gamma^2 \\
&= \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \beta\gamma + \gamma\alpha)
\end{aligned}
$$

E.g. 3 The cubic equation $2x^3 + 3x^2 + 4x + 5 = 0$ has roots α, β and γ. Find the values of:

(a) $\alpha^2 + \beta^2 + \gamma^2$ (b) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

Working: (a)

$$\alpha + \beta + \gamma = -\frac{b}{a} = -\frac{3}{2}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} = \frac{4}{2} = 2$$

$$\alpha\beta\gamma = -\frac{d}{a} = -\frac{5}{2}$$

$$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$$

$$= \left(-\frac{3}{2}\right)^2 - 2 \times 2$$

$$= -\frac{7}{4}$$

(b)

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma}{\alpha\beta\gamma} + \frac{\gamma\alpha}{\alpha\beta\gamma} + \frac{\alpha\beta}{\alpha\beta\gamma}$$

$$= \frac{\alpha\beta + \beta\gamma + \gamma\alpha}{\alpha\beta\gamma}$$

$$= \frac{2}{-\frac{5}{2}}$$

$$= -\frac{4}{5}$$

E.g. 4 Write down the equation whose roots are α, β and γ where $\alpha + \beta + \gamma = 6$, $\alpha\beta + \alpha\gamma + \beta\gamma = 11$ and $\alpha\beta\gamma = 6$. Hence solve the equation.

Working:

$\alpha + \beta + \gamma = 6$ ***(change the sign)***

$\alpha\beta + \alpha\gamma + \beta\gamma = 11$ ***(don't change the sign)***

$\alpha\beta\gamma = 6$ ***(change the sign)***

The equation is $x^3 - 6x^2 + 11x - 6 = 0$

Let $f(x) = x^3 - 6x^2 + 11x - 6$

$f(1) = 1 - 6 + 11 - 6 = 0$ $\Rightarrow$ $x - 1$ is a root

$x^3 - 6x^2 + 11x - 6 = (x - 1)(ax^2 + bx + c)$

By inspection, $a = 1$ and $c = 6$

Equating coefficients: 'x': $11 = c - b$

Since $c = -6$ $b = -5$

$x^3 - 6x^2 + 11x - 6 = (x - 1)(x^2 - 5x + 6)$

$x^2 - 5x + 6 = (x - 2)(x - 3)$

The roots are 1, 2 and 3.

Video: Sum and product of roots (cubic)
Video: Sum and product of roots (quartic)

Solutions to Starter and E.g.s

Exercise
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