Determinant and Inverse of 3 by 3 Matrices

Starter

1. Find the matrix $\mathbf{X}$ such that $\begin{pmatrix}1 & 3\\ 2 & 4\end{pmatrix}\mathbf{X}^{-1} = \begin{pmatrix}-2 & 9\\ -2 & 14\end{pmatrix}$.

Working: Post-multiply by $\mathbf{X}$:

$$\begin{pmatrix}1 & 3\\ 2 & 4\end{pmatrix}\mathbf{X}^{-1}\mathbf{X} = \begin{pmatrix}-2 & 9\\ -2 & 14\end{pmatrix}\mathbf{X}$$

But $\mathbf{X}^{-1}\mathbf{X} = \mathbf{I}$ $\quad \begin{pmatrix}1 & 3\\ 2 & 4\end{pmatrix} = \begin{pmatrix}-2 & 9\\ -2 & 14\end{pmatrix}\mathbf{X}$

Pre-multiply by $\begin{pmatrix}-2 & 9\\ -2 & 14\end{pmatrix}^{-1}$:

$$\begin{pmatrix}-2 & 9\\ -2 & 14\end{pmatrix}^{-1}\begin{pmatrix}1 & 3\\ 2 & 4\end{pmatrix} = \begin{pmatrix}-2 & 9\\ -2 & 14\end{pmatrix}^{-1}\begin{pmatrix}-2 & 9\\ -2 & 14\end{pmatrix}\mathbf{X}$$

But $\begin{pmatrix}-2 & 9\\ -2 & 14\end{pmatrix}^{-1}\begin{pmatrix}-2 & 9\\ -2 & 14\end{pmatrix} = \mathbf{I}$ and $\mathbf{IX} = \mathbf{X}$.

So $\therefore \mathbf{X} = \begin{pmatrix}-2 & 9\\ -2 & 14\end{pmatrix}^{-1}\begin{pmatrix}1 & 3\\ 2 & 4\end{pmatrix}$

$$\begin{pmatrix}-2 & 9\\ -2 & 14\end{pmatrix}^{-1} = \frac{1}{-10}\begin{pmatrix}14 & -9\\ 2 & -2\end{pmatrix}$$

$$\therefore \mathbf{X} = \frac{1}{-10}\begin{pmatrix}14 & -9\\ 2 & -2\end{pmatrix}\begin{pmatrix}1 & 3\\ 2 & 4\end{pmatrix} = \begin{pmatrix}\frac{2}{5} & -\frac{3}{5}\\ \frac{1}{5} & \frac{1}{5}\end{pmatrix}$$

2. If $\mathbf{A}$ is the matrix $\begin{pmatrix}7 & 4\\ 6 & 3\end{pmatrix}$, show that $\mathbf{A}^2 - 10\mathbf{A} - 3\mathbf{I} = 0$. Hence find $\mathbf{A}^{-1}$.

Hint: Remember $\mathbf{AA}^{-1} = \mathbf{I}$

Working: Rearranging $\mathbf{A}^2 - 10\mathbf{A} - 3I = 0$: $\quad \mathbf{A}^2 - 10\mathbf{A} = 3\mathbf{I}$

Factorise the LHS: $\quad \mathbf{A}(\mathbf{A} - 10\mathbf{I}) = 3\mathbf{I}$

N.B. It is not $\mathbf{A}(\mathbf{A} - 10) = 3\mathbf{I}$ because 10 is a number not a matrix.

Divide by 3: $\quad \frac{1}{3}\mathbf{A}(\mathbf{A} - 10\mathbf{I}) = \mathbf{I}$

So $\mathbf{A}^{-1} = \frac{1}{3}(\mathbf{A} - 10\mathbf{I})$

$$= \frac{1}{3}\left(\begin{pmatrix}7 & 4\\ 6 & 3\end{pmatrix} - 10\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right)$$

$$\mathbf{A}^{-1} = \frac{1}{3}\begin{pmatrix}-3 & 4\\ 6 & -7\end{pmatrix}$$

E.g. 1 Find the determinant of the matrix $\begin{pmatrix} 1 & -2 & 3 \\ 1 & -1 & 2 \\ -2 & 4 & -5 \end{pmatrix}$.

Working:

$$\begin{vmatrix} 1 & -2 & 3 \\ 1 & -1 & 2 \\ -2 & 4 & -5 \end{vmatrix} = 1\begin{vmatrix} -1 & 2 \\ 4 & -5 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 2 \\ -2 & -5 \end{vmatrix} + 3\begin{vmatrix} 1 & -1 \\ -2 & 4 \end{vmatrix}$$
$$= -3 - 2 + 6$$

$$\begin{vmatrix} 1 & -2 & 3 \\ 1 & -1 & 2 \\ -2 & 4 & -5 \end{vmatrix} = 11$$

E.g. 2 Using the shortcuts above, find the determinants of these matrices:

(a) $\begin{pmatrix} 1 & -2 & 3 \\ 1 & -1 & 2 \\ -2 & 4 & -5 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ -2 & -2 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & -1 & 3 \\ 1 & 2 & 1 \\ 3 & -4 & 5 \end{pmatrix}$

Working: (a) Transpose: $\begin{vmatrix} 1 & -2 & 3 \\ 1 & -1 & 2 \\ -2 & 4 & -5 \end{vmatrix} = \begin{vmatrix} 1 & 1 & -2 \\ -2 & -1 & 4 \\ 3 & 2 & -5 \end{vmatrix}$

$R_1 + R_2$: $\begin{vmatrix} -1 & 0 & 2 \\ -2 & -1 & 4 \\ 3 & 2 & -5 \end{vmatrix}$

$$\begin{vmatrix} -1 & 0 & 2 \\ -2 & -1 & 4 \\ 3 & 2 & -5 \end{vmatrix} = -1\begin{vmatrix} -1 & 4 \\ 2 & -5 \end{vmatrix} + 2\begin{vmatrix} -2 & -1 \\ 3 & 2 \end{vmatrix}$$
$$= 3 + 2 \times (-1) = 1$$

(b) $2R_1 + R_3$: $\begin{vmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ -2 & -2 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ -1 & 0 & 1 \\ -2 & -2 & 0 \end{vmatrix}$

$$\begin{vmatrix} 0 & 0 & 1 \\ -1 & 0 & 1 \\ -2 & -2 & 0 \end{vmatrix} = 1\begin{vmatrix} -1 & 0 \\ -2 & -2 \end{vmatrix} = 2$$

(c) $2R_1 + R_2$: $\begin{vmatrix} 2 & -1 & 3 \\ 1 & 2 & 1 \\ 3 & -4 & 5 \end{vmatrix} = \begin{vmatrix} 5 & 0 & 7 \\ 1 & 2 & 1 \\ 3 & -4 & 5 \end{vmatrix}$

$R_3 + 2R_2$: $\begin{vmatrix} 5 & 0 & 7 \\ 1 & 2 & 1 \\ 3 & -4 & 5 \end{vmatrix} = \begin{vmatrix} 5 & 0 & 7 \\ 1 & 2 & 1 \\ 5 & 0 & 7 \end{vmatrix}$

Two rows are the same so the determinant is zero.

E.g. 3 Find the inverse of the matrix $\mathbf{M} = \begin{pmatrix} 4 & 1 & -1 \\ 3 & -1 & 2 \\ 4 & 0 & -3 \end{pmatrix}$

Working: det $\mathbf{M} = 25$

Matrix of cofactors is $\mathbf{C} = \begin{pmatrix} 3 & 17 & 4 \\ 3 & -8 & 4 \\ 1 & -11 & -7 \end{pmatrix}$ so $\mathbf{C}^T = \begin{pmatrix} 3 & 3 & 1 \\ 17 & -8 & -11 \\ 4 & 4 & -7 \end{pmatrix}$

$$\mathbf{M}^{-1} = \frac{1}{25}\begin{pmatrix} 3 & 3 & 1 \\ 17 & -8 & -11 \\ 4 & 4 & -7 \end{pmatrix}$$

E.g. 4 Find the inverse of the matrix $\begin{pmatrix} 1 & -2 & 3 \\ 1 & -1 & 2 \\ -2 & 4 & -5 \end{pmatrix}$.

Hint: You have already found its determinant in ***E.g. 2***

Working: The determinant is 1.

Matrix of cofactors is $\mathbf{C} = \begin{pmatrix} -3 & 1 & 2 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}$ so $\mathbf{C}^T = \begin{pmatrix} -3 & 2 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}$

The inverse matrix is $\frac{1}{1}\begin{pmatrix} -3 & 2 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 2 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}$

E.g. 5 Use your calculator to find the inverse of these matrices:

(a) $\begin{pmatrix} 1 & -2 & -3 \\ 2 & -1 & -4 \\ 3 & -3 & -5 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & -2 & -1 \\ 2 & 1 & 5 \\ 3 & -2 & 3 \end{pmatrix}$

Working: (a) $\begin{pmatrix} -\frac{7}{6} & -\frac{1}{6} & \frac{5}{6} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$

(b) $\begin{pmatrix} \frac{13}{2} & 4 & -\frac{9}{2} \\ \frac{9}{2} & 3 & -\frac{7}{2} \\ -\frac{7}{2} & -2 & \frac{5}{2} \end{pmatrix}$

E.g. 6 Using your calculator, solve for $\mathbf{X}$ the equation $\begin{pmatrix} 1 & 3 & 2 \\ 4 & 0 & -1 \\ 2 & 3 & -3 \end{pmatrix}\mathbf{X} = \begin{pmatrix} 0 & -6 & 3 \\ 21 & 6 & -9 \\ -9 & 5 & -4 \end{pmatrix}$.

Working: $\begin{pmatrix} 1 & 3 & 2 \\ 4 & 0 & -1 \\ 2 & 3 & -3 \end{pmatrix}^{-1} = \frac{1}{57}\begin{pmatrix} 3 & 15 & -3 \\ 10 & -7 & 9 \\ 12 & 3 & 4 \end{pmatrix}$

Pre-multiply by $\begin{pmatrix} 1 & 3 & 2 \\ 4 & 0 & -1 \\ 2 & 3 & -3 \end{pmatrix}^{-1}$:

$$\mathbf{X} = \frac{1}{57}\begin{pmatrix} 3 & 15 & -3 \\ 10 & -7 & 9 \\ 12 & 3 & 4 \end{pmatrix}\begin{pmatrix} 0 & -6 & 3 \\ 21 & 6 & -9 \\ -9 & 5 & -4 \end{pmatrix}$$

$$\mathbf{X} = \begin{pmatrix} 6 & 1 & -2 \\ -4 & -1 & 1 \\ 3 & -2 & 1 \end{pmatrix}$$

Video: Determinant of 3 by 3 matrices
Video A: Inverse of 3 by 3 matrices
Video B: Inverse of 3 by 3 matrices
Video: Using a calculator to find the determinant and/or the inverse of matrices

Solutions to Starter and E.g.s

Exercise
p27 1E Qu 1i, 2i, 3i, 4i, 5-9