# Discrete Uniform Distributions

**Starter**

1. **(Review of last lesson)**

The discrete random $X$ has p.d.f. give by $P(X = x) = \dfrac{3x + 1}{22}$ for $x = 0, 1, 2, 3$. Find:

(a) $E(X)$ and $E(X^2)$ (b) $\text{Var}(X)$
(c) $E(3X - 2)$ (d) $E(2X^2 + 4X - 3)$
(e) $\text{Var}(4X)$ (f) $\text{Var}(5X + 7)$

***Working:*** (a)

$$P(X = 0) = \frac{3 \times 0 + 1}{22} = \frac{1}{22}$$
$$P(X = 1) = \frac{3 \times 1 + 1}{22} = \frac{4}{22}$$
$$P(X = 2) = \frac{3 \times 2 + 1}{22} = \frac{7}{22}$$
$$P(X = 3) = \frac{3 \times 3 + 1}{22} = \frac{10}{22}$$
$$E(X) = 1 \times \frac{4}{22} + 2 \times \frac{7}{22} + 3 \times \frac{10}{22} = \frac{48}{22} = \frac{24}{11}$$
$$E(X^2) = 1^2 \times \frac{4}{22} + 2^2 \times \frac{7}{22} + 3^2 \times \frac{10}{22} = \frac{122}{22} = \frac{61}{11}$$

(b) $$\text{Var}(X) = E(X^2) - E^2(X) = \frac{61}{11} - \left(\frac{24}{11}\right)^2 = \frac{95}{121}$$

(c) $$E(3X - 2) = 3E(X) - 2 = 3 \times \frac{24}{11} - 2 = \frac{50}{11}$$

(d)
$$E(2X^2 + 4X - 3) = 2E(X^2) + 4E(X) - 3$$
$$= 2 \times \frac{61}{11} + 4 \times \frac{24}{11} - 3$$
$$= 16\frac{9}{11} = \frac{185}{11}$$

(e) $$\text{Var}(4X) = 4^2 \times \text{Var}(X) = 16 \times \frac{95}{121} = \frac{1520}{121}$$

(f) $$\text{Var}(5X + 7) = 5^2 \times \text{Var}(X) = 25 \times \frac{95}{121} = \frac{2375}{121}$$

2. Without writing out all the terms, calculate how many terms are in the following linear sequences?

(a) $5, 7, 9, ..., 37$

(b) $3, 7, 11, ..., 31$

(c) $a, a+d, a+2d, \ldots, b$

***Working:*** (a) Number of terms $= \dfrac{37-5}{2} + 1 = 17$

(b) Number of terms $= \dfrac{31-3}{4} + 1 = 8$

(c) Number of terms $= \dfrac{b-a}{d} + 1$

***E.g. 1*** Let $X$ be a discrete uniform distribution which can take the values $6, 9, 12, \ldots, 60$. Find $P(X = 48)$.

***Working:*** There are $20$ multiples of $3$ between $0$ and $60$ so $X$ can take $19$ values and since each probability is the same $P(X = 48) = \dfrac{1}{19}$

***Alternatively:*** Number of terms $= \dfrac{60-6}{3} + 1 = 19$ terms

So $P(X = 48) = \dfrac{1}{19}$

***E.g. 2*** The random variable $X$ can take the values $a, a+d, a+2d, \ldots, b$. Find an expression for $P(X = x)$ in terms of $a, b$ and $d$.

***Working:*** Number of terms $= \dfrac{b-a}{d} + 1 = \dfrac{b-a}{d} + \dfrac{d}{d} = \dfrac{b-a+d}{d}$

$\therefore P(X = x) = \dfrac{d}{b-a+d}$

***E.g. 3*** Given that the sum of the first $n$ integers is $\frac{1}{2}n(n+1)$ and that the sum of the first $n$ square numbers is $\frac{1}{6}n(n+1)(2n+1)$, find the mean and variance of $X \sim \mathrm{U}(n)$.

***Hint:*** remember if the values of $x$ are not stated, assume $x = 1,2,3,...,n$

***Working:***

| $x$ : | 1 | 2 | 3 | … | $n$ |
|---|---|---|---|---|---|
| $P(X=x)$ : | $\frac{1}{n}$ | $\frac{1}{n}$ | $\frac{1}{n}$ | … | $\frac{1}{n}$ |

$$\begin{aligned}\text{Mean} &= (1+2+3+\ldots+n)\times\frac{1}{n}\\ &= \frac{1}{2}n(n+1)\times\frac{1}{n}\\ &= \frac{1}{2}(n+1)\end{aligned}$$

$$\begin{aligned}\text{Variance} &= E(X^2)-E^2(X)\\ &= (1^2+2^2+3^2+\ldots+n^2)\times\frac{1}{n}-\left(\frac{n+1}{2}\right)^2\\ &= \frac{1}{6}n(n+1)(2n+1)\times\frac{1}{n}-\left(\frac{n+1}{2}\right)^2\\ &= \frac{1}{6}(n+1)(2n+1)-\left(\frac{n+1}{2}\right)^2\\ &= \frac{1}{12}(n+1)\left[2(2n+1)-3(n+1)\right]\\ &= \frac{1}{12}(n+1)(n-1)\\ &= \frac{1}{12}(n^2-1)\end{aligned}$$

***E.g. 4*** Find the mean and variance for $X \sim \mathrm{U}(31)$

***Working:***

$$\text{Mean} = \frac{n+1}{2} = \frac{31+1}{2} = 16$$

$$\text{Variance} = \frac{1}{12}(31^2-1) = 80$$

***E.g. 5*** An icosahedron is rolled (i.e. a $20-$sided dice). Find the mean and standard deviation.

***N.B.*** Icosahedron is pronounced “i-coss-a-hee-dron”.

***Working:***

$$X \sim \mathrm{U}(20)$$

$$\text{Mean} = \frac{n+1}{2} = \frac{20+1}{2} = 10.5$$

$$\text{Variance} = \frac{1}{12}(20^2-1) = 33.25$$

So standard deviation $= \sqrt{33.25} = \frac{\sqrt{133}}{2} = 5.77$ (3 s.f.)

Video: Discrete uniform distributions

Solutions to Starter and E.g.s

**Exercise**
p29 2C Qu 1i, 2-8, (9-10 red)