Division and Complex Conjugates

Starter

1. **(Review of last lesson)** Given that $(1+5\text{i})A - 2B = 3 + 7\text{i}$, find A and B if:
   (a) A and B are both real
   (b) A and B are both complex.

Working: (a) $(1+5\text{i})A - 2B = 3 + 7\text{i}$

Equating real and imaginary parts:

Re: $A - 2B = 3$

Im: $5A = 7 \quad \Rightarrow \quad A = \frac{7}{5}$

$B = -\frac{4}{5}$

(b) $(1+5\text{i})A - 2B = 3 + 7\text{i}$

Equating real and imaginary parts:

Re: $-5A = 3 \quad \Rightarrow \quad A = -\frac{3}{5}$

Im: $A - 2B = 7 \quad \Rightarrow \quad -\frac{3}{5} - 2B = 7$

$B = -\frac{19}{5}$

So $A = -\frac{3}{5}\text{i}$ and $B = -\frac{19}{5}\text{i}$

2. **(Review of last lesson)** Find the square root of $-7 + 24\text{i}$.

Working Let $z = a + b\text{i}$ be such that $z = \sqrt{-7+24\text{i}}$, where a and b are real

i.e. $a + b\text{i} = \sqrt{-7+24\text{i}}$

Squaring both sides gives $a^2 - b^2 + 2ab\text{i} = -7 + 24\text{i}$

Equating real and imaginary parts:

Re: $a^2 - b^2 = -7$

Im: $2ab = 24 \quad ab = 12 \quad b = \frac{12}{a}$

Substituting: $a^2 - \left(\frac{12}{a}\right)^2 = -7$

$a^4 - 144 = -7a^2$

$a^4 + 7a^2 - 144 = 0$

$(a^2 - 9)(a^2 + 16) = 0$

$a^2 = 9 \quad \text{or} \quad a^2 = -16$

$\therefore a = \pm 3$ since $a^2 = -16$ gives imaginary values

When $a = 3$, $b = 4$

When $a = -3$, $b = -4$

The square roots of $-7 + 24\text{i}$ are $3 + 4\text{i}$ and $-3 - 4\text{i}$.

3. If $p = 2 + 3\text{i}$ and $q = 2 - 3\text{i}$, express the following in the form $a + b\text{i}$, where a and b are real numbers.

(a) $p + q$ (b) $p - q$ (c) pq (d) $(p + q)(p - q)$
(e) $p^2 - q^2$ (f) $p^2 + q^2$ (g) $(p + q)^2$ (h) $(p - q)^2$

Working: (a) 4 (b) 6i

(c) $pq = (2 + 3\text{i})(2 - 3\text{i}) = 4 - 6\text{i} + 6\text{i} + 9 = 13$

(d) $(p + q)(p - q) = 4 \times 6\text{i} = 24\text{i}$

(e) $p^2 - q^2 = (2 + 3\text{i})^2 - (2 - 3\text{i})^2 = 4 + 12\text{i} - 9 - (4 - 12\text{i} - 9) = 24\text{i}$

(f) $p^2 + q^2 = (2 + 3\text{i})^2 + (2 - 3\text{i})^2 = 4 + 12\text{i} - 9 + 4 - 12\text{i} - 9 = -10$

(g) $(p + q)^2 = 4^2 = 16$

(h) $(p - q)^2 = (6\text{i})^2 = -36$

E.g. 1 Let $z = x + y\text{i}$ where x and y are real numbers. Find:

(a) z^* (b) $z + z^*$ (c) $z - z^*$ (d) zz^*

Working: (a) $z^* = x - y\text{i}$

(b) $z + z^* = x + y\text{i} + x - y\text{i} = 2x = 2\text{Re}(z)$

(c) $z - z^* = x + y\text{i} - x - y\text{i} = 2y = 2\text{Im}(z)$

(d) $zz^* = (x + y\text{i})(x - y\text{i}) = x^2 - xy\text{i} + xy\text{i} + y^2 = x^2 + y^2$

E.g. 2 Write down z^* given that $z =$:

(a) $2 + 4\text{i}$ (b) $3 - 6\text{i}$ (c) $-5 + 2\text{i}$
(d) $2\text{i} - 4$ (e) 6 (f) $-3\text{i} + 7$

Working: (a) $2 - 4\text{i}$ (b) $3 + 6\text{i}$

(c) $-5 - 2\text{i}$ (d) $-4 - 2\text{i}$

(e) 6 (f) $7 + 3\text{i}$

E.g. 3 (a) Rationalise the denominator of $\dfrac{5 + \sqrt{3}}{4 + \sqrt{3}}$.

(b) Using a similar method to (a), express $\dfrac{5 + 3\text{i}}{4 + 3\text{i}}$ in the form $\dfrac{a + b\text{i}}{c}$ where a, b and c are real numbers.

Working: (a) $\dfrac{5+\sqrt{3}}{4+\sqrt{3}} = \dfrac{5+\sqrt{3}}{4+\sqrt{3}} \times \dfrac{4-\sqrt{3}}{4-\sqrt{3}}$

$$= \frac{20-5\sqrt{3}+4\sqrt{3}-3}{16-3}$$

$$= \frac{17-\sqrt{3}}{13}$$

(b) $\dfrac{5+3\mathrm{i}}{4+3\mathrm{i}} = \dfrac{5+3\mathrm{i}}{4+3\mathrm{i}} \times \dfrac{4-3\mathrm{i}}{4-3\mathrm{i}}$

$$= \frac{20-15\mathrm{i}+15\mathrm{i}+9}{16+9}$$

$$= \frac{29-3\mathrm{i}}{25}$$

E.g. 4 Express $\dfrac{2-7\mathrm{i}}{1+2\mathrm{i}}$ in the form $a+b\mathrm{i}$ where a and b are real.

Working: $\dfrac{2-7\mathrm{i}}{1+2\mathrm{i}} = \dfrac{2-7\mathrm{i}}{1+2\mathrm{i}} \times \dfrac{1-2\mathrm{i}}{1-2\mathrm{i}}$

$$= \frac{2-4\mathrm{i}-7\mathrm{i}-14}{1+4}$$

$$= -\frac{12}{5} - \frac{11}{5}\mathrm{i}$$

E.g. 5 Given that a and b are real and $(a+b\mathrm{i})(3+4\mathrm{i}) = 3-4\mathrm{i}$ find a and b.

Working: $(a+b\mathrm{i})(3+4\mathrm{i}) = 3-4\mathrm{i} \quad \Rightarrow \quad a+b\mathrm{i} = \dfrac{3-4\mathrm{i}}{3+4\mathrm{i}}$

$$a+b\mathrm{i} = \frac{3-4\mathrm{i}}{3+4\mathrm{i}} \times \frac{3-4\mathrm{i}}{3-4\mathrm{i}}$$

$$= \frac{9-12\mathrm{i}-12\mathrm{i}-16}{9+16}$$

$$= -\frac{7}{25} - \frac{24}{25}\mathrm{i}$$

$a = -\dfrac{7}{25}$ and $b = -\dfrac{24}{25}$

Video: Complex conjugates
Video: Division of complex numbers

Solutions to Starter and E.g.s

Exercise
p117 4B Qu 1i, 2i, 3-6, 7i, 8-14