Finding an Equation with Given Roots

Starter

1. **(Review of last lesson)**

Let the roots of the equation $x^3 - 6x + 2 = 0$ be α, β and γ. Find the value of:

(a) $4\alpha + 4\beta + 4\gamma$ (b) $\alpha^2 + \beta^2 + \gamma^2$

(c) $(\beta - \gamma)^2 + (\gamma - \alpha)^2 + (\alpha - \beta)^2$ (d) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$

Working: (a)

$$\alpha + \beta + \gamma = -\frac{b}{a} = 0$$
$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a} = -6$$
$$\alpha\beta\gamma = -\frac{d}{a} = -2$$
$$4\alpha + 4\beta + 4\gamma = 4(\alpha + \beta + \gamma) = 0$$

(b)

$$\begin{aligned}\alpha^2 + \beta^2 + \gamma^2 &= (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)\\ &= 0^2 - 2 \times (-6)\\ &= 12\end{aligned}$$

(c)

$$\begin{aligned}(\beta - \gamma)^2 + (\gamma - \alpha)^2 + (\alpha - \beta)^2 &= \beta^2 - 2\beta\gamma + \gamma^2 + \gamma^2 - 2\gamma\alpha + \alpha^2 + \alpha^2 - 2\alpha\beta + \beta^2\\ &= 2(\alpha^2 + \beta^2 + \gamma^2) - 2(\alpha\beta + \beta\gamma + \gamma\alpha)\\ &= 2 \times 12 - 2 \times (-6)\\ &= 36\end{aligned}$$

(d)

$$\begin{aligned}\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} &= \frac{\beta\gamma}{\alpha\beta\gamma} + \frac{\gamma\alpha}{\alpha\beta\gamma} + \frac{\alpha\beta}{\alpha\beta\gamma}\\ &= \frac{\alpha\beta + \beta\gamma + \gamma\alpha}{\alpha\beta\gamma}\\ &= \frac{-6}{-2}\\ &= 3\end{aligned}$$

(FP1 p140)

2. The quadratic equation $x^2 + 5x + 7 = 0$ has roots α and β. Without calculating α and β, find an equation with roots 2α and 2β.

Working: From $x^2 + 5x + 7 = 0$, $\alpha + \beta = -5$ and $\alpha\beta = 7$

For new equation: sum of root $= 2\alpha + 2\beta = 2(\alpha + \beta) = -10$

product of roots $= 2\alpha \times 2\beta = 4\alpha\beta = 28$

So the new equation is $x^2 + 10x + 28 = 0$

E.g. 1 The equation $x^2 + 2x + 5 = 0$ has roots α and β. Use the "roots method" to find equations with integer coefficients which have the following roots.

(a) 3α and 3β (b) $\alpha + 1$ and $\beta + 1$ (c) $\frac{1}{\alpha}$ and $\frac{1}{\beta}$

Working: (a) From $x^2 + 2x + 5 = 0$, $\alpha + \beta = -2$ and $\alpha\beta = 5$
For the new equation:
Sum of root $= 3\alpha + 3\beta = 3(\alpha + \beta) = -6$ ***change sign***
Product of roots $= 3\alpha \times 3\beta = 9\alpha\beta = 45$ ***do not change sign***
So the new equation is $x^2 + 6x + 45 = 0$

(b) For the new equation:
Sum of root $= \alpha + 1 + \beta + 1 = \alpha + \beta + 2 = 0$
Product of roots $= (\alpha + 1)(\beta + 1) = \alpha\beta + \alpha + \beta + 1 = 4$
So the new equation is $x^2 + 4 = 0$

(c) For the new equation:
Sum of root $= \frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = -\frac{2}{5}$ ***change sign***
Product of roots $= \frac{1}{\alpha} \times \frac{1}{\beta} = \frac{1}{\alpha\beta} = \frac{1}{5}$ ***do not change sign***
So the new equation is $x^2 + \frac{2}{5}x + \frac{1}{5} = 0$
To get integer coefficients, multiply by 5: $5x^2 + 2x + 1 = 0$

E.g. 2 The equation $2x^3 - 5x^2 + 3x + 4 = 0$ has roots α, β and γ. Use the "roots method" to find equations with integer coefficients which have the following roots.
(a) 3α, 3β and 3γ (b) $\alpha - 1$, $\beta - 1$ and $\gamma - 1$

Working: (a) $\alpha + \beta + \gamma = -\frac{b}{a} = \frac{5}{2}$

$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a} = \frac{3}{2}$

$\alpha\beta\gamma = -\frac{d}{a} = -2$

$3\alpha + 3\beta + 3\gamma = 3(\alpha + \beta + \gamma) = \frac{15}{2}$ ***change sign***

$3\alpha \times 3\beta + 3\beta \times 3\gamma + 3\gamma \times 3\alpha = 9(\alpha\beta + \beta\gamma + \gamma\alpha)$
$= \frac{27}{2}$ ***do not change sign***

$3\alpha \times 3\beta \times 3\gamma = 27\alpha\beta\gamma = -54$ ***change sign***

So the new equation is $x^3 - \frac{15}{2}x^2 + \frac{27}{2}x + 54 = 0$

Multiply by 2: $2x^3 - 15x^2 + 27x + 108 = 0$

(b) $(\alpha-1)+(\beta-1)+(\gamma-1)=\frac{5}{2}-3=-\frac{1}{2}$ ***change sign***

$$(\alpha-1)(\beta-1)+(\beta-1)(\gamma-1)+(\gamma-1)(\alpha-1)=\alpha\beta-\alpha-\beta+1+\beta\gamma-\beta-\gamma+1+\gamma\alpha-\gamma-\alpha+1$$
$$=\alpha\beta+\beta\gamma+\gamma\alpha-2(\alpha+\beta+\gamma)+3$$
$$=\frac{3}{2}-2\times\frac{5}{2}+3$$
$$=-\frac{1}{2}$$ ***do not change sign***

$$(\alpha-1)(\beta-1)(\gamma-1)=\alpha\beta\gamma-\alpha\beta-\beta\gamma-\gamma\alpha+\alpha+\beta+\gamma-1$$
$$=\alpha\beta\gamma-(\alpha\beta+\beta\gamma+\gamma\alpha)+\alpha+\beta+\gamma-1$$
$$=-2-\frac{3}{2}+\frac{5}{2}-1$$
$$=-2$$ ***change sign***

So the new equation is $x^3+\frac{1}{2}x^2-\frac{1}{2}x+2=0$

Multiply by 2: $2x^3+x^2-x+4=0$

Other questions can be asked that require you to deduce the relationship between coefficients.

E.g. 3 One root of the equation $ax^2+bx+c=0$ is the reciprocal of the other. Prove that $c=a$.

Working: Let the roots be α and $\frac{1}{\alpha}$.

Then product of roots is 1 so $\frac{c}{a}=1$

i.e. $c=a$

E.g. 4 Find the quadratic equation $ax^2+bx+c=0$, one root is twice the other. Prove that $2b^2=9ac$.

Working: Let the roots be α and 2α.

Sum of roots is $3\alpha=-\frac{b}{a}$ $\Rightarrow$ $\alpha=-\frac{b}{3a}$

Product of roots is $2\alpha^2=\frac{c}{a}$ $\Rightarrow$ $2\left(-\frac{b}{3a}\right)^2=\frac{c}{a}$

$\frac{2b^2}{9a^2}=\frac{c}{a}$ $\Rightarrow$ $2b^2=9ac$

Video: Finding equation given roots
Video: Finding equation given roots example

Solutions to Starter and E.g.s

Exercise
p159 5E Qu 1i, 2i, 3-9, (10-12 red)