Further transformations in 2-D

Starter

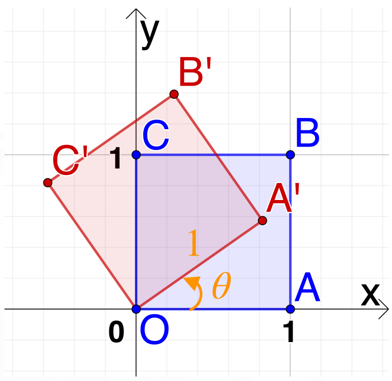
1. The matrix $\begin{pmatrix} 2 & 4 \\ 1 & 4 \end{pmatrix}$ transforms the unit square from $OABC$ to $OA'B'C'$. Find the matrix which transforms the unit square to $OC'B'A'$.

2. The matrix $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ causes an enlargement, scale factor 3 about the origin.
The matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ causes a reflection in the $x-$axis.
Find the matrix that causes an enlargement, scale factor 3 followed by a reflection in the $x-$axis

3. Find the matrix that causes a stretch in $x-$direction, scale factor 2.

Rotations

E.g. 1 By considering the unit square, deduce the matrix that gives an anti-clockwise rotation of θ about the origin?



Working: From the diagram:

A': $x-$coordinate $1\cos\theta = \cos\theta$
$y-$coordinate $1\sin\theta = \sin\theta$

C': $x-$coordinate $-1\sin\theta = -\sin\theta$
$y-$coordinate $1\cos\theta = \cos\theta$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}$

The matrix is $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

E.g. 2 By considering the unit square, deduce the matrix that causes a shear, factor 2, parallel to:
(a) the $x-$axis (b) the $y-$axis?

Working: (a) Parallel to the $x-$axis $\Rightarrow$ $y-$coordinate is unchanged

$(1,\ 0)$ is on the $x-$axis so is invariant i.e. $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 0 + 2 \times 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

So the matrix is $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

(b) Parallel to the $y-$axis $\Rightarrow$ $x-$coordinate is unchanged

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 + 2 \times 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$(0,\ 1)$ is on the $y-$axis so is invariant i.e. $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

So the matrix is $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$.

E.g. 3 A triangle undergoes a shear, $y-$axis invariant, mapping $(1,\ 0) \to (1,\ 7)$. The point $(3,\ 5)$ is a vertex of the triangle before the shear. Find the new coordinates of the vertex.

Working: $y-$axis invariant $\Rightarrow$ $x-$coordinate is unchanged

$(1,\ 0) \to (1,\ 7)$ $\Rightarrow$ the factor is 7

$$\begin{pmatrix} 3 \\ 5 \end{pmatrix} \to \begin{pmatrix} 3 \\ 5 + 7 \times 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 26 \end{pmatrix}$$

The new coordinates of the vertex are $(3,\ 26)$

E.g. 4 Given that a 2 by 2 matrix $\mathbf{M}$ transforms $\begin{pmatrix} 2 \\ 1 \end{pmatrix} \to \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \to \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, find $\mathbf{M}$.

Working: $\mathbf{M}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\mathbf{M}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$$\therefore \mathbf{M}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}$$

Post-multiply by $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1}$: $\mathbf{M} = \begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -2 & 5 \\ 1 & 1 \end{pmatrix}$

E.g. 5 Given that a 2 by 2 matrix $\mathbf{M}$ transforms $\begin{pmatrix} 3 \\ 2 \end{pmatrix} \to \begin{pmatrix} 12 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix} \to \begin{pmatrix} -1 \\ -3 \end{pmatrix}$, find $\mathbf{M}$.

Working: $\mathbf{M}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ 1 \end{pmatrix}$ and $\mathbf{M}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$

$$\therefore \mathbf{M}\begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 12 & -1 \\ 1 & -3 \end{pmatrix}$$

Post-multiply by $\begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}^{-1}$:

$$\mathbf{M} = \begin{pmatrix} 12 & -1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix}$$

Video: Matrix rotations

Video: Combination of matrix transformations

Video: Determinants and matrix transformations

Solutions to Starter and E.g.s

Exercise

p82 3C Qu 1ac, 2ac, 3-8