Geometric Distributions

Starter

1. (Review of last lesson) If the random variable X is such that $X \sim \mathrm{B}(10, p)$, where $p < \dfrac{1}{2}$ and $\mathrm{Var}(X) = \dfrac{15}{8}$, find:

(a) the value of p,
(b) $E(X)$ and
(b) $P(X = 2)$.

Working:

(a) $X \sim B(10, p) \quad \Rightarrow \quad n = 10$

$$\mathrm{Var}(X) = \frac{15}{8} \Rightarrow \quad 10p(1-p) = \frac{15}{8}$$
$$16(p - p^2) = 3$$
$$16p^2 - 16p + 3 = 0$$
$$(4p - 3)(4p - 1) = 0$$

Since $p < \dfrac{1}{2}$, $p = \dfrac{1}{4}$

(b) $E(X) = np = 10 \times \dfrac{1}{4} = 2.5$

(c) $P(X = 2) = {}^{10}C_2 \times 0.25^2 \times 0.75^8 = 0.282$ (3 s.f.)

2. A sports magazine is giving away signed photos of famous swimmers. It randomly places 15 photos in every 100 magazines. Let X be the number of magazines you purchase until you get a signed photo (i.e. the first $X - 1$ magazines do not have signed photos in).

(a) Find the probability of getting a signed photo in the 3rd magazine you purchase but not in the first 2 i.e. find $P(X = 3)$.
(b) Explain what $P(X = 20)$ means in this context.
(c) Calculate $P(X = 10)$.
(d) Write down an expression for $P(X = x)$.
(e) Calculate the probability that you need to buy 1, 2, 3 or 4 magazines to get a signed photo.

Working:

(a) $P(\text{photo}) = \dfrac{15}{100} = 0.15$

$$P(X = 3) = 0.85^2 \times 0.15 = 0.108375 = \frac{867}{8000}$$

(b) Buy 19 magazines with no photo, 20th has a photo.

(c) ***No photo in the first 9 magazines, but photo in the 10th.***

$P(X = 10) = 0.85^9 \times 0.15 = 0.0347$ (3 s.f.)

(d) $P(X = x) = 0.85^{x-1} \times 0.15$

(e) $P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) = 0.478$

E.g. 1 A child has a spinner with five equal sections, numbered 1 to 5. To start a game, they must spin a 4 or a 5. The child spins until a 4 or 5 is spun. Calculate the probability that they need:

(a) exactly 4 spins
(b) at most 2 spins
(c) at least 7 spins
(d) at most 12 spins
(e) How many spins are required to be $99\,\%$ sure of getting a 4 or 5?

Working:

(a) $X \sim$ Geo(0.4)

Exactly 4 spins $\equiv$ 3 fails followed by success on the 4th spin

$$P(X=4) = 0.6^3 \times 0.4 = 0.0864 = \frac{54}{625}$$

(b)
$$\begin{aligned} P(X \le 2) &= P(X=1) + P(X=2) \\ &= 0.4 + 0.6 \times 0.4 \\ &= 0.64 = \frac{16}{25} \end{aligned}$$

(c) At least 7 spins is $P(X \ge 7) = P(X=7) + P(X=8) + \ldots$
However, it is easiest to realise that $P(X \ge 7)$ is equivalent to having had 6 fails.

$$\therefore P(X \ge 7) = 0.6^6 = 0.46656 = \frac{729}{15625}$$

(d) $P(X \le 12) = P(X=1) + P(X=2) + \ldots + P(X=12)$ but it is easier to consider the complementary event.
i.e. $P(X \le 12) = 1 - P(X \ge 13)$
Using similar thinking to (c), $P(X \ge 13)$ is equivalent to 12 fails.
$\therefore P(X \le 12) = 1 - 0.6^{12} = 0.998$ (3 s.f.)

(e) We need to find x such that $P(X \le x) > 0.99$.

Using similar thinking to (d): $1 - P(X \ge x) > 0.99$

$$1 - 0.6^x > 0.99$$
$$0.6^x < 0.01$$

Solve using logs: $\ln 0.6^x < \ln 0.01$

$$x \ln 0.6 < \ln 0.01$$

$\ln 0.6 < 0$ ***so inequality changes direction:*** $x > \dfrac{\ln 0.01}{\ln 0.6}$

$$x > 9.015$$

So 10 spins are required.

E.g. 2 Let $X \sim$ Geo(p).

(a) Copy and complete the table.

x:	1	2	3	4	x
$P(X=x)$:					

(b) Find $P(X \ge x)$

(c) Find $P(X \le x)$

Working: (a)

x:	1	2	3	4	x
$P(X=x)$:	p	$(1-p)p$	$(1-p)^2p$	$(1-p)^3p$	$(1-p)^{x-1}p$

(b) $P(X \geq x) = P(X=x) + P(X=x+1) + P(X=x+2) + \dots$
Consider $P(X \geq 5) = P(X=5) + P(X=6) + \dots$
$P(X \geq 5) = 1 - P(X \leq 4)$
$= 1 - P(X=1) - P(X=2) - P(X=3) - P(X=4)$
$= 1 - p - (1-p)p - (1-p)^2p - (1-p)^3p$
$= (1-p)\left[(1-p-(1-p)p-(1-p)p\right]$
$= (1-p)^2\left[(1-p-(1-p)p\right]$
$= (1-p)^3\left[1-p\right]$
$= (1-p)^4$
$\equiv$ 4 fails
So it is easier to think of $P(X \geq x)$ as $x-1$ fails.
$\therefore P(X \geq x) = (1-p)^{x-1}$

Alternatively, if you have done sum to infinity of a geometric progression:

$P(X \geq x) = P(X=x) + P(X=x+1) + P(X=x+2) + \dots$
$= (1-p)^{x-1}p + (1-p)^xp + (1-p)^{x+1}p + \dots$
which is a sum to infinity with $a = (1-p)^{x-1}p$ and $r = 1-p$

$$S_\infty = \frac{a}{1-r} = \frac{(1-p)^{x-1}p}{1-(1-r)} = \frac{(1-p)^{x-1}p}{p} = (1-p)^{x-1}$$

$\therefore P(X \geq x) = (1-p)^{x-1}$

(c) $P(X \leq x) = P(X=1) + P(X=2) + \dots + P(X=x)$
It is easier to consider the complementary event and then use similar thinking to (b).
$P(X \leq x) = 1 - P(X \geq x+1)$
But $P(X \geq x+1)$ is equivalent to x fails.
So $P(X \leq x) = 1 - P(x \text{ fails}) = 1 - (1-p)^x$

E.g. 3 It is given that the random variable, T, which can take values $1, 2, 3, \dots$ has a geometric distribution and $P(T=1) = 0.15$. Calculate:
(a) $P(T \geq 8)$ (b) $P(T > 11)$ (c) $P(T \leq 9)$ (d) $P(T < 15)$.
Give your answers to 4 s.f.

Working: (a) Since $P(T=1) = 0.15$, $p = 0.15$ so $1-p = 0.85$
$P(T \geq 8) = P(7 \text{ fails})$
$= 0.85^7$
$= 0.3206$ (4 s.f.)

(b) $P(T > 11) = P(T \geq 12)$
$= P(11 \text{ fails})$
$= 0.85^{11}$
$= 0.1673$ (4 s.f.)

(c) $P(T \leq 9) = 1 - P(T \geq 10)$
$= 1 - P(9 \text{ fails})$

$$= 1 - 0.85^9$$
$$= 0.7683 \text{ (4 s.f.)}$$

(d) $P(T < 15) = P(T \leq 14)$
$$= 1 - P(T \geq 15)$$
$$= 1 - P\left(14 \text{ fails}\right)$$
$$= 1 - 0.85^{14}$$
$$= 0.8972 \text{ (4 s.f.)}$$

Video: Geometric distributions
Video: Least trials before success

Solutions to Starter and E.g.s

Exercise
p33 2E Qu 1i, 2i, 3-9, (10 red)