Geometric Distributions

Starter

1. (Review of last lesson) If the random variable X is such that $X \sim \text{B}(10, p)$, where $p < \frac{1}{2}$ and $\text{Var}(X) = \frac{15}{8}$, find:
   (a) the value of p,
   (b) $E(X)$ and
   (b) $P(X = 2)$.

Working: (a) $X \sim B(10, p) \quad \Rightarrow \quad n = 10$

$$\text{Var}(X) = \frac{15}{8} \quad \Rightarrow \quad 10p(1-p) = \frac{15}{8}$$

$$16(p - p^2) = 3$$

$$16p^2 - 16p + 3 = 0$$

$$(4p-3)(4p-1) = 0$$

Since $p < \frac{1}{2}$, $p = \frac{1}{4}$

(b) $E(X) = np = 10 \times \frac{1}{4} = 2.5$

(c) $P(X = 2) = {}^{10}C_2 \times 0.25^2 \times 0.75^8 = 0.282$ (3 s.f.)

2. A sports magazine is giving away signed photos of famous swimmers. It randomly places 15 photos in every 100 magazines. Let X be the number of magazines you purchase until you get a signed photo (i.e. the first $X - 1$ magazines do not have signed photos in).
   (a) Find the probability of getting a signed photo in the 3rd magazine you purchase but not in the first 2 i.e. find $P(X = 3)$.
   (b) Explain what $P(X = 20)$ means in this context.
   (c) Calculate $P(X = 10)$.
   (d) Write down an expression for $P(X = x)$.
   (e) Calculate the probability that you need to buy 1, 2, 3 or 4 magazines to get a signed photo.

Working: (a) $P(\text{photo}) = \frac{15}{100} = 0.15$

$$P(X = 3) = 0.85^2 \times 0.15 = 0.108375 = \frac{867}{8000}$$

(b) Buy 19 magazines with no photo, 20th has a photo.

(c) ***No photo in the first 9 magazines, but photo in the 10th.***
$P(X = 10) = 0.85^9 \times 0.15 = 0.0347$ (3 s.f.)

(d) $P(X = x) = 0.85^{x-1} \times 0.15$

(e) $P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) = 0.478$

E.g. 1 A child has a spinner with five equal sections, numbered 1 to 5. To start a game, they must spin a 4 or a 5. The child spins until a 4 or 5 is spun. Calculate the probability that they need:

(a) exactly 4 spins
(b) at most 2 spins
(c) at least 7 spins
(d) at most 12 spins
(e) How many spins are required to be $99\,\%$ sure of getting a 4 or 5?

Working:

(a) $X \sim \text{Geo}(0.4)$

Exactly 4 spins $\equiv 3$ fails followed by success on the 4th spin

$$P(X = 4) = 0.6^3 \times 0.4 = 0.0864 = \frac{54}{625}$$

(b)
$$\begin{aligned} P(X \leq 2) &= P(X = 1) + P(X = 2) \\ &= 0.4 + 0.6 \times 0.4 \\ &= 0.64 = \frac{16}{25} \end{aligned}$$

(c) At least 7 spins is $P(X \geq 7) = P(X = 7) + P(X = 8) + \ldots$
However, it is easiest to realise that $P(X \geq 7)$ is equivalent to having had 6 fails.

$$\therefore P(X \geq 7) = 0.6^6 = 0.46656 = \frac{729}{15625}$$

(d) $P(X \leq 12) = P(X = 1) + P(X = 2) + \ldots + P(X = 12)$ but it is easier to consider the complementary event.
i.e. $P(X \leq 12) = 1 - P(X \geq 13)$
Using similar thinking to (c), $P(X \geq 13)$ is equivalent to 12 fails.
$\therefore P(X \leq 12) = 1 - 0.6^{12} = 0.998$ (3 s.f.)

(e) We need to find x such that $P(X \leq x) > 0.99$.

Using similar thinking to (d): $1 - P(X \geq x) > 0.99$

$$1 - 0.6^x > 0.99$$
$$0.6^x < 0.01$$

Solve using logs: $\ln 0.6^x < \ln 0.01$

$$x \ln 0.6 < \ln 0.01$$

$\ln 0.6 < 0$ ***so inequality changes direction:*** $x > \dfrac{\ln 0.01}{\ln 0.6}$

$$x > 9.015$$

So 10 spins are required.

E.g. 2 Let $X \sim \text{Geo}(p)$.

(a) Copy and complete the table.

x:	1	2	3	4	x
$P(X = x)$:					

(b) Find $P(X \geq x)$

(c) Find $P(X \leq x)$

Working: (a)

x:	1	2	3	4	x
$P(X=x)$:	p	$(1-p)p$	$(1-p)^2p$	$(1-p)^3p$	$(1-p)^{x-1}p$

(b) $P(X \geq x) = P(X=x) + P(X=x+1) + P(X=x+2) + \ldots$
But it is easier to think of $P(X \geq x)$ as there having been $x-1$ fails.
$\therefore P(X \geq x) = (1-p)^{x-1}$

(c) $P(X \leq x) = P(X=1) + P(X=2) + \ldots + P(X=x)$
But it is easier to consider the complementary event and then use similar thinking to (b)
$P(X \leq x) = 1 - P(X \geq x+1)$
$P(X \geq x+1)$ is equivalent to x fails
So $P(X \leq x) = 1 - P(x \text{ fails}) = 1-(1-p)^x$

E.g. 3 It is given that the random variable, T, which can take values $1, 2, 3, \ldots$ has a geometric distribution and $P(T=1)=0.15$. Calculate:
(a) $P(T \geq 8)$ (b) $P(T > 11)$ (c) $P(T \leq 9)$ (d) $P(T < 15)$.
Give your answers to 4 s.f.

Working: (a) Since $P(T=1)=0.15$, $p=0.15$ so $1-p=0.85$
$$P(T \geq 8) = P(7 \text{ fails}) = 0.85^7 = 0.3206 \text{ (4 s.f.)}$$

(b)
$$P(T > 11) = P(T \geq 12) = P(11 \text{ fails}) = 0.85^{11} = 0.1673 \text{ (4 s.f.)}$$

(c)
$$P(T \leq 9) = 1 - P(T \geq 10) = 1 - P(9 \text{ fails}) = 1 - 0.85^9 = 0.7683 \text{ (4 s.f.)}$$

(d)
$$P(T < 15) = P(T \leq 14) = 1 - P(T \geq 15) = 1 - P(14 \text{ fails}) = 1 - 0.85^{14} = 0.8972 \text{ (4 s.f.)}$$

Video: Geometric distributions
Video: Least trials before success

Solutions to Starter and E.g.s

Exercise
p33 2E Qu 1i, 2i, 3-9, (10 red)