Induction and divisibility

Starter

1. Prove that $\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ 3^n - 1 & 3^n \end{pmatrix}$ for any positive integer using induction.

Working: ***(Proposition)***

Let $P(n)$ be the proposition that $\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ 3^n - 1 & 3^n \end{pmatrix}$.

(Prove the basic case)

When $n = 1$: $\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}^1 = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$ and

$\begin{pmatrix} 1 & 0 \\ 3^1 - 1 & 3^1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$

Therefore $P(1)$ is true.

(Inductive step — n replaced by k)

Assume that $P(k)$ is true i.e. $\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}^k = \begin{pmatrix} 1 & 0 \\ 3^k - 1 & 3^k \end{pmatrix}$

(Inductive step — multiply both sides by $\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$)

$$\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & 0 \\ 3^k - 1 & 3^k \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$$

(Algebraic manipulation until RHS is k replace by $k+1$)

Multiplying the matrices:

$$\begin{aligned} \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}^{k+1} &= \begin{pmatrix} 1 & 0 \\ 3^k - 1 & 3^k \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 3^k - 1 + 2 \times 3^k & 3 \times 3^k \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 3 \times 3^k - 1 & 3^{k+1} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 3^{k+1} - 1 & 3^{k+1} \end{pmatrix} \end{aligned}$$

(Completion)

But this is $P(k)$ with k replaced by $k + 1$.

Therefore, if $P(k)$ is true, then $P(k + 1)$ is also true.

$P(1)$ is true and if $P(k)$ is true, then $P(k + 1)$ is true. Using the principle of mathematical induction, $P(n)$ is true for all positive integers.

E.g. 1 Prove that, for positive integers n, $8^n - 3^n$ is always divisible by 5.

Working:

(Proposition)
Let $P(n)$ be the proposition that $8^n - 3^n$ is always divisible by 5.
(Prove the basic case)
When $n = 1$, $8^1 - 3^1 = 5$ which is divisible by 5
Therefore $P(1)$ is true.
(Inductive step — n replaced by k)
Assume that $P(k)$ is true i.e. $8^k - 3^k$ is always divisible by 5
So $8^k - 3^k = 5M$ where M is a positive integer
(Inductive step — consider the next term)
$P(k+1)$ is the term $8^{k+1} - 3^{k+1}$.
(Inductive step — manipulation to show this is also divisible by 5)
$P(k+1) = 8 \times 8^k - 3 \times 3^k$.
But $8^k = 3^k + 5M$
$P(k+1) = 8 \times (3^k + 5M) - 3 \times 3^k = 5 \times 3^k + 40M = 5(3^k + 8M)$
which is divisible by 5.
(Completion)
But this is $P(k)$ with k replaced by $k+1$.
Therefore, if $P(k)$ is true, then $P(k+1)$ is also true.
$P(1)$ is true and if $P(k)$ is true, then $P(k+1)$ is true. Using the principle of mathematical induction, $P(n)$ is true for all positive integers.

Alternative inductive manipulation steps
(Inductive step — manipulation to show this is also divisible by 5)

$$\begin{aligned} P(k+1) &= 8 \times 8^k - 3 \times 3^k \\ &= (5+3)8^k - 3 \times 3^k \\ &= 5 \times 8^k + 3 \times 8^k - 3 \times 3^k \\ &= 5 \times 8^k + 3(8^k - 3^k) \\ &= 5 \times 8^k + 3P(k) \end{aligned}$$

Since we assume $P(k)$ is true, both terms are divisible by 5.
Hence $P(k+1)$ is divisible by 5

E.g. 2 Prove by mathematical induction that $6^n + 4$ is divisible by 5 for positive integers values of n.

Working:

(Proposition)
Let $P(n)$ be the proposition that $6^n + 4$ is divisible by 5 when $n \in \mathbb{Z}^+$.
(Prove the basic case)
When $n = 1$, $6^1 + 4 = 5$ which is divisible by 5
Therefore $P(1)$ is true.
(Inductive step — n replaced by k)
Assume that $P(k)$ is true i.e. $6^k + 4$ is always divisible by 5 for $k \in \mathbb{Z}^+$
So $6^k + 4 = 5M$ where M is a positive integer
(Inductive step — consider the next term)
$P(k+1)$ is the term $6^{k+1} + 4$.
(Inductive step — manipulation to show this is also divisible by 5)
$P(k+1) = 6^{k+1} + 4 = 6 \times 6^k + 4$.
But $6^k + 4 = 5M$ so $6^k = 5M - 4$
$P(k+1) = 6 \times (5M - 4) + 4 = 30M - 20 = 5(6M - 4)$
which is divisible by 5.
(Completion)
But this is $P(k)$ with k replaced by $k + 1$.
Therefore, if $P(k)$ is true, then $P(k+1)$ is also true.
$P(1)$ is true and if $P(k)$ is true, then $P(k+1)$ is true. Using the principle of mathematical induction, $P(n)$ is true for all positive integers.

Alternative inductive manipulation steps
(Inductive step — manipulation to show this is also divisible by 5)

$$
\begin{aligned}
P(k+1) &= 6^{k+1} + 4 \\
&= 6 \times 6^k + 4 \\
&= (5+1)6^k + 4 \\
&= 5 \times 6^k + 6^k + 4 \\
&= 5 \times 6^k + P(k)
\end{aligned}
$$

Since we assume $P(k)$ is true, both terms are divisible by 5.
Hence $P(k+1)$ is divisible by 5

E.g. 3 Prove that if $n \geq 1$ is a positive integer, then $13^n - 6^n$ is divisible by 7.

Working:

(Proposition)
Let $P(n)$ be the proposition that $13^n - 6^n$ is divisible by 7 when $n \in \mathbb{Z}^+$.
(Prove the basic case)
When $n = 1$, $13^1 - 6^1$ which is divisible by 7
Therefore $P(1)$ is true.
(Inductive step — n replaced by k)
Assume that $P(k)$ is true i.e. $13^k - 6^k$ is always divisible by 7 for $k \in \mathbb{Z}^+$
So $13^n - 6^n = 7M$ where M is a positive integer .
(Inductive step — consider the next term)
$P(k+1)$ is the term $13^{n+1} - 6^{n+1}$.
(Inductive step — manipulation to show this is also divisible by 7)
$P(k+1) = 13^{n+1} - 6^{n+1} = 13 \times 13^n - 6^{n+1}$.
But $13^n - 6^n = 7M$ so $13^n = 6^n + 7M$

$$\begin{aligned} P(k+1) &= 13(6^n + 7M) - 6^{n+1} \\ &= 7 \times 6^n + 7M \\ &= 7(6^n + M) \end{aligned}$$

which is divisible by 7.
(Completion)
But this is $P(k)$ with k replaced by $k+1$.
Therefore, if $P(k)$ is true, then $P(k+1)$ is also true.
$P(1)$ is true and if $P(k)$ is true, then $P(k+1)$ is true. Using the principle of mathematical induction, $P(n)$ is true for all positive integers.

Alternative inductive manipulation steps
(Inductive step — manipulation to show this is also divisible by 7)

$$\begin{aligned} P(k+1) &= 13^{n+1} - 6^{n+1} \\ &= 13 \times 13^n - 6^{n+1} \\ &= (7+6) \times 13^n - 6^{n+1} \\ &= 7 \times 13^n + 6 \times 13^n - 6 \times 6^n \\ &= 7 \times 13^n + 6(13^n - 6^n) \\ &= 7 \times 13^n + 6P(k) \end{aligned}$$

Since we assume $P(k)$ is true, both terms are divisible by 7.
Hence $P(k+1)$ is divisible by 7

Video: Proof by induction (divisibility)

Solutions to Starter and E.g.s

Exercise
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