Induction and inequalities

Starter

1. Prove that $4^n + 5^n + 6^n$ is divisible by 15 by mathematical induction, when n is an odd positive integer.

Working:

(Proposition)
Let $P(n)$ be the proposition that $4^n + 5^n + 6^n$ is divisible by 15 when n is an odd, positive integer.
(Prove the basic case)
When $n = 1$, $4^1 + 5^1 + 6^1 = 15$ which is divisible by 15
Therefore $P(1)$ is true.
(Inductive step — n replaced by k)
Assume that $P(k)$ is true i.e. $4^k + 5^k + 6^k$ is divisible by 15 when k is an odd, positive integer.
So $4^k + 5^k + 6^k = 15M$ where M is a positive integer .
(Inductive step — consider the next term)
Since n is odd, 2 needs to be added to the powers.
$P(k+2)$ is the term $4^{k+2} + 5^{k+2} + 6^{k+2}$.
(Inductive step — manipulation to show this is also divisible by 15)

$$P(k+2) = 4^{k+2} + 5^{k+2} + 6^{k+2}$$
$$= 16 \times 4^k + 5^{k+2} + 6^{k+2}$$

But $4^k + 5^k + 6^k = 15M$ so $4^k = 15M - 5^k - 6^k$

$$P(k+2) = 16 \times (15M - 5^k - 6^k) + 25 \times 5^k + 36 \times 6^k$$
$$= 240M + 9 \times 5^k + 20 \times 6^k$$
$$= 240M + 45 \times 5^{k-1} + 120 \times 6^{k-1}$$

which is divisible by 15 since 240, 45 and 120 are all divisible by 15.
(Completion)
But this is $P(k)$ with k replaced by $k+1$.
Therefore, if $P(k)$ is true, then $P(k+1)$ is also true.
$P(1)$ is true and if $P(k)$ is true, then $P(k+1)$ is true. Using the principle of mathematical induction, $P(n)$ is true for all positive integers.

Alternative inductive manipulation steps
(Inductive step — manipulation to show this is also divisible by 15)

$$P(k+2) = 4^{k+2} + 5^{k+2} + 6^{k+2}$$
$$= 16 \times 4^k + 25 \times 5^k + 36 \times 6^k$$
$$= 16 \times 4^k + (16+9) \times 5^k + (16+20) \times 6^k$$
$$= 16(4^k + 5^k + 6^k) + 9 \times 5^k + 20 \times 6^k$$
$$= 16P(k) + 45 \times 5^{k-1} + 120 \times 6^{k-1}$$

Since we assume $P(k)$ is true, all terms are divisible by 15.
Hence $P(k+1)$ is divisible by 15

E.g. 1 Prove that $2^n > n$ for all natural numbers n.

Working:

(Proposition)
Let $P(n)$ be the proposition that $2^n > n$ for all natural numbers n.
(Prove the basic case)
When $n = 1$: $2^1 = 2 > 1$ which is true
Therefore $P(1)$ is true.
(Inductive step)
Assume that $P(k)$ is true i.e. $2^k > k$
(Inductive step — consider the next term)
Need to prove $P(k+1)$ is true i.e. $2^{k+1} > k+1$
(Inductive step — algebraic manipulation)

$P(k+1)$: $2^{k+1} = 2 \times 2k$
$> 2 \times k$ assuming $P(k)$ is true
$= k + k$
$\geq k + 1$ since $k \geq 1$

(Completion)
But this is $P(k)$ with k replaced by $k+1$.
Therefore, if $P(k)$ is true, then $P(k+1)$ is also true.
$P(1)$ is true and if $P(k)$ is true, then $P(k+1)$ is true. Using the principle of mathematical induction, $P(n)$ is true for all positive integers.

E.g. 2 Prove that $4^{n-1} > n^2$ for $n \geq 3$.

Working:

(Proposition)
Let $P(n)$ be the proposition that $4^{n-1} > n^2$ for $n \geq 3$.
(Prove the basic case)
When $n = 3$: $4^{3-1} = 16 > 3^2 = 9$ which is true.
Therefore $P(3)$ is true.
(Inductive step)
Assume that $P(k)$ is true i.e. $4^{k-1} > k^2$
(Inductive step — consider the next term)
Need to prove $P(k+1)$ is true i.e. $4^k > (k+1)^2$
(Inductive step — algebraic manipulation)

$P(k+1)$: $4^k = 4 \times 4^{k-1}$
$> 4 \times k^2$ assuming $P(k)$ is true
$= k^2 + 2k^2 + k^2$
$> k^2 + 2k + 1$ since $2k^2 > 2k$ & $k^2 > 1$ for $k \geq 3$
$= (k+1)^2$

(Completion)
But this is $P(k)$ with k replaced by $k+1$.
Therefore, if $P(k)$ is true, then $P(k+1)$ is also true.
$P(1)$ is true and if $P(k)$ is true, then $P(k+1)$ is true. Using the principle of mathematical induction, $P(n)$ is true for all positive integers.

N.B. $(n+1)! = (n+1) \times n!$

E.g. 3 Prove that $n! > 2^n$ for $n \geq 4$.

Working:

(Proposition)
Let $P(n)$ be the proposition that $n! > 2^n$ for $n \geq 4$.
(Prove the basic case)
When $n = 4$: $4! = 24 > 2^4 = 16$ which is true.
Therefore $P(4)$ is true.
(Inductive step)
Assume that $P(k)$ is true i.e. $k! > 2^k$ for $k \geq 4$.
(Inductive step — consider the next term)
Need to prove $P(k+1)$ is true i.e. $(k+1)! > 2^{k+1}$
(Inductive step — algebraic manipulation)

$P(k+1)$: $(k+1)! = (k+1) \times k!$

$> (k+1) \times 2^k$ assuming $P(k)$ is true

$> 2 \times 2^k$ since $k+1 > 2$ for $k \geq 4$

$= 2^{k+1}$

(Completion)
But this is $P(k)$ with k replaced by $k+1$.
Therefore, if $P(k)$ is true, then $P(k+1)$ is also true.
$P(1)$ is true and if $P(k)$ is true, then $P(k+1)$ is true. Using the principle of mathematical induction, $P(n)$ is true for all positive integers.

Video A: Proof by induction (inequalities)
Video B: Proof by induction (inequalities)

Solutions to Starter and E.g.s

Exercise
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