Induction and matrices

Starter

1. Given that $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, $\mathbf{k} \times \mathbf{r} = \mathbf{p}$ and $\mathbf{r} \times \mathbf{p} = \mathbf{k}$, where a, b and c are constants, find an expression connecting a and b and the value of c.

Working: $\mathbf{k} \times \mathbf{r} = \mathbf{p}$: $\quad \mathbf{p} = \mathbf{k} \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ a & b & c \end{vmatrix} = -b\mathbf{i} + a\mathbf{j}$

$\mathbf{r} \times \mathbf{p} = \mathbf{k}$: $\quad \mathbf{r} \times \mathbf{p} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ -b & a & 0 \end{vmatrix} = -ac\mathbf{i} - bc\mathbf{j} + (a^2 + b^2)\mathbf{k} \equiv \mathbf{k}$

Equating components:

$\mathbf{i}$: $\quad ac = 0$

$\mathbf{j}$: $\quad bc = 0$

$\mathbf{k}$: $\quad a^2 + b^2 = 1$

If $a = 0$, $b^2 = 1$ $\quad \Rightarrow \quad$ $c = 0$ so a does not need to equal zero

Therefore, $a^2 + b^2 = 1$ and $c = 0$.

E.g. 1 Let $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$. Prove that $\mathbf{A}^n = \begin{pmatrix} 1 & 0 \\ 5n & 1 \end{pmatrix}$ for all natural numbers n.

Working: ***(Proposition)***

Let $P(n)$ be the proposition that if $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$ then $\mathbf{A}^n = \begin{pmatrix} 1 & 0 \\ 5n & 1 \end{pmatrix}$

(Prove the basic case)

When $n = 1$, $\mathbf{A}^1 = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 5 \times 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$

Therefore $P(1)$ is true.

(Inductive step — n replaced by k)

Assume that $P(k)$ is true i.e. if $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$ then $\mathbf{A}^k = \begin{pmatrix} 1 & 0 \\ 5k & 1 \end{pmatrix}$

(Inductive step — multiply both sides by $\mathbf{A}$ to get $\mathbf{A}^{k+1}$)

$\mathbf{A}^{k+1} = \begin{pmatrix} 1 & 0 \\ 5k & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$

(Algebraic manipulation until RHS is k replace by $k + 1$)

Multiplying the matrices: $\quad \mathbf{A}^{k+1} = \begin{pmatrix} 1 & 0 \\ 5k + 5 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 5(k+1) & 1 \end{pmatrix}$

(Completion)

But this is $P(k)$ with k replaced by $k + 1$.

Therefore, if $P(k)$ is true, then $P(k + 1)$ is also true.

$P(1)$ is true and if $P(k)$ is true, then $P(k + 1)$ is true. Using the principle of mathematical induction, $P(n)$ is true for all positive integers.

E.g. 2 Let $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Calculate the matrices $\mathbf{A}^2$ and $\mathbf{A}^3$. Make a conjecture about the matrix $\mathbf{A}^n$ and prove it by induction.

Working: $\mathbf{A}^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $\mathbf{A}^2 = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$ $\Rightarrow$ conjecture is $\mathbf{A}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$

(Proposition)

Let $P(n)$ be the proposition that $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ then $\mathbf{A}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$.

(Prove the basic case)

When $n = 1$, $\mathbf{A}^1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Therefore $P(1)$ is true.

(Inductive step — n replaced by k)

Assume that $P(k)$ is true i.e. if $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ then $\mathbf{A}^k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$

(Inductive step — multiply both sides by $\mathbf{A}$ to get $\mathbf{A}^{k+1}$)

$\mathbf{A}^{k+1} = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

(Algebraic manipulation until RHS is k replace by $k+1$)

Multiplying the matrices: $\mathbf{A}^{k+1} = \begin{pmatrix} 1 & k+1 \\ 0 & 1 \end{pmatrix}$

(Completion)

But this is $P(k)$ with k replaced by $k + 1$.
Therefore, if $P(k)$ is true, then $P(k+1)$ is also true.
$P(1)$ is true and if $P(k)$ is true, then $P(k+1)$ is true. Using the principle of mathematical induction, $P(n)$ is true for all positive integers.

E.g. 3 Prove that $\begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}^n = \begin{pmatrix} 1-3n & 9n \\ -n & 1+3n \end{pmatrix}$ using induction.

Working:

(Proposition)

Let $P(n)$ be the proposition that $\begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}^n = \begin{pmatrix} 1-3n & 9n \\ -n & 1+3n \end{pmatrix}$.

(Prove the basic case)

When $n = 1$: $\begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}^1 = \begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}$ and

$$\begin{pmatrix} 1-3\times 1 & 9\times 1 \\ -\times 1 & 1+3\times 1 \end{pmatrix} = \begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}$$

Therefore $P(1)$ is true.

(Inductive step — n replaced by k)

Assume that $P(k)$ is true i.e. $\begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}^k = \begin{pmatrix} 1-3k & 9k \\ -k & 1+3k \end{pmatrix}$

(Inductive step — multiply both sides by $\begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}$)

$$\begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}^{k+1} = \begin{pmatrix} 1-3k & 9k \\ -k & 1+3k \end{pmatrix}\begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}$$

(Algebraic manipulation until RHS is k replace by $k+1$)

Multiplying the matrices:

$$\begin{aligned} \begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}^{k+1} &= \begin{pmatrix} 1-3k & 9k \\ -k & 1+3k \end{pmatrix}\begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix} \\ &= \begin{pmatrix} -2(1-3k)-9k & 9(1-3k)+36k \\ 2k-1-3k & -9k+4(1+3k) \end{pmatrix} \\ &= \begin{pmatrix} -2-3k & 9+9k \\ -k-1 & 3k+4 \end{pmatrix} \\ &= \begin{pmatrix} 1-3(k+1) & 9(k+1) \\ -(k+1) & 1+3(k+1) \end{pmatrix} \end{aligned}$$

(Completion)

But this is $P(k)$ with k replaced by $k+1$.
Therefore, if $P(k)$ is true, then $P(k+1)$ is also true.
$P(1)$ is true and if $P(k)$ is true, then $P(k+1)$ is true. Using the principle of mathematical induction, $P(n)$ is true for all positive integers.

Video: Proof by induction (matrices)

Solutions to Starter and E.g.s

Exercise
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