Intersection of two lines

Starter

1. Convert $\mathbf{r} = (2 - 3\lambda)\mathbf{i} + (4 + 2\lambda)\mathbf{j} + \lambda\mathbf{k}$ to Cartesian form:

Working:

$$x = 2 - 3\lambda \quad \Rightarrow \quad \lambda = \frac{x - 2}{-3}$$

$$y = 4 + 2\lambda \quad \Rightarrow \quad \lambda = \frac{y - 4}{2}$$

$$z = \lambda \quad \Rightarrow \quad \lambda = z$$

The Cartesian equation is $\frac{2 - x}{3} = \frac{y - 4}{2} = z = (\lambda)$

2. Write down the Cartesian equation of the line that passes through the point $(5, -4, 0)$ and is parallel to the vector $\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$.

Working:

The vector equation is $\mathbf{r} = \begin{pmatrix} 5 \\ -4 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$

$$x = 5 + \lambda \quad \Rightarrow \quad \lambda = x - 5$$

$$y = -4$$

$$z = 3\lambda \quad \Rightarrow \quad \lambda = \frac{1}{3}z$$

The Cartesian equation is $x - 5 = \frac{z}{3}$ and $y = -4$

3. Find the coordinates of the point of intersection between the lines with equations $\mathbf{r} = \begin{pmatrix} -3 \\ -1 \\ 7 \end{pmatrix} + s\begin{pmatrix} -4 \\ 2 \\ 5 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} -13 \\ -7 \\ 8 \end{pmatrix} + t\begin{pmatrix} 7 \\ 2 \\ -3 \end{pmatrix}$.

Working:

Equate the lines: $\begin{pmatrix} -3 \\ -1 \\ 7 \end{pmatrix} + s\begin{pmatrix} -4 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} -13 \\ -7 \\ 8 \end{pmatrix} + t\begin{pmatrix} 7 \\ 2 \\ -3 \end{pmatrix}$

Equating components:

$\mathbf{i}$: $-3 - 4s = -13 + 7t \quad \Rightarrow \quad 4s + 7t = 10$

$\mathbf{j}$: $-1 + 2s = -7 + 2t \quad \Rightarrow \quad 2s - 2t = -6$

$\mathbf{k}$: $7 + 5s = 8 - 3t \quad \Rightarrow \quad 5s + 3t = 1$

Solving the $\mathbf{i}$ and $\mathbf{j}$ equations simultaneously: $s = -1, t = 2$

Check these values work in the $\mathbf{k}$ equation: $5 \times (-1) + 3 \times 2 = 1$ ✓

Substitute either the $s-$ or the $t-$value in the relevant equation:

Either… $s = -1$ $\quad \mathbf{r} = \begin{pmatrix} -3 \\ -1 \\ 7 \end{pmatrix} - 1\begin{pmatrix} -4 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$

…or… $t = 2$: $\quad \mathbf{r} = \begin{pmatrix} -13 \\ -7 \\ 8 \end{pmatrix} + 2\begin{pmatrix} 7 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$

The point of intersection between the lines is $(1, -3, 2)$.

E.g. 1 Find the coordinates of the point of intersection between the lines with equations

(a) $\mathbf{r} = \begin{pmatrix} 5 \\ 4 \\ 18 \end{pmatrix} + s\begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} -12 \\ 3 \\ 19 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix}$

(b) $\mathbf{r} = \begin{pmatrix} 4 \\ -4 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} -19 \\ -22 \\ 1 \end{pmatrix} + t\begin{pmatrix} 5 \\ 6 \\ -1 \end{pmatrix}$

Working: (a) Equate the lines: $\begin{pmatrix} 5 \\ 4 \\ 18 \end{pmatrix} + s\begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} -12 \\ 3 \\ 19 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix}$

Equating components:

$\mathbf{i}$: $5 + 3s = -12 + 2t \quad \Rightarrow \quad 3s - 2t = -17$

$\mathbf{j}$: $4 - s = 3 + t \quad \Rightarrow \quad s + t = 1$

$\mathbf{k}$: $18 + 5s = 19 - 4t \quad \Rightarrow \quad 5s + 4t = 1$

Solving the $\mathbf{i}$ and $\mathbf{j}$ equations simultaneously: $s = -3, t = 4$

Check values work in the $\mathbf{k}$ equation: $5 \times (-3) + 4 \times 4 = 1$ ✓

Substitute either the $s-$ or the $t-$value in the relevant equation:

Either… $s = -3$ $\quad \mathbf{r} = \begin{pmatrix} 5 \\ 4 \\ 18 \end{pmatrix} - 3\begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} -4 \\ 7 \\ 3 \end{pmatrix}$

…or… $t = 4$: $\quad \mathbf{r} = \begin{pmatrix} -12 \\ 3 \\ 19 \end{pmatrix} + 4\begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} = \begin{pmatrix} -4 \\ 7 \\ 3 \end{pmatrix}$

The point of intersection between the lines is $(-4, 7, 3)$.

(b) Equate the lines: $\begin{pmatrix} 4 \\ -4 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -19 \\ -22 \\ 1 \end{pmatrix} + t\begin{pmatrix} 5 \\ 6 \\ -1 \end{pmatrix}$

Equating components:

$\mathbf{i}$: $4 - s = -19 + 5t \quad \Rightarrow \quad s + 5t = 23$

$\mathbf{j}$: $-4 + 2s = -22 + 6t \quad \Rightarrow \quad 2s - 6t = -18$

$\mathbf{k}$: $s = 1 - t \quad \Rightarrow \quad s + t = 1$

Solving the $\mathbf{i}$ and $\mathbf{j}$ equations simultaneously: $s = 3, t = 4$

Check values work in the $\mathbf{k}$ equation: $3 + 4 = 1$ ✘

Since the equation for $\mathbf{k}$ is inconsistent with the values found from the $\mathbf{i}$ and $\mathbf{j}$ equations, there is no point of intersection.

Video: Intersecting and skew lines

Parallel, intersecting and skew lines EQ

Solutions to Starter and E.g.s

Exercise

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