Introduction to Complex Numbers

Starter

1. (Review of GCSE material)
Solve the equation $x^2 - 2x + 5 = 0$ using the quadratic formula.

Working:
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{2 \pm \sqrt{2^2 - 4 \times 1 \times 5}}{2 \times 1}$$
$$x = \frac{2 \pm \sqrt{-16}}{2}$$

E.g. 1 Write down the value of these powers of i: i^2 i^3 i^4 i^5 .

Working: i^2 = −1 i^3 = −i i^4 = 1 i^5 = i

E.g. 2 Let $z_1 = 3 - 4\text{i}$ and let $z_2 = 2 + 5\text{i}$. Find:
(a) $z_1 + z_2$ (b) $z_1 - z_2$ (c) $z_1 \times z_2$

Working: (a) $z_1 + z_2 = 3 - 4\text{i} + 2 + 5\text{i} = 5 + \text{i}$

(b) $z_1 - z_2 = 3 - 4\text{i} - 2 + 5\text{i} = 1 - 9\text{i}$1 − 9i

(c) $z_1 \times z_2 = (3 - 4\text{i})(2 + 5\text{i}) = 6 + 15\text{i} - 8\text{i} + 20 = 26 + 7\text{i}$
Remember, $\text{i} \times \text{i} = -1$ so $-\text{i} \times \text{i} = 1$

E.g. 3 Let $z = a + b\text{i}$ where a and b are real numbers. If $z = 0$, find the values of a and b.

Working If $z = 0$ then $a + b\text{i} = 0$ and $a = -b\text{i}$
Squaring both sides gives $a^2 = -b^2$
These can only be equal if $a = 0$ and $b = 0$

E.g. 4 Find the real numbers x and y such that $x + 4y + xy\text{i} = 12 - 16\text{i}$.

Working: Equating real parts: $x + 4y = 12$

Equating imaginary parts: $xy = -16 \quad \Rightarrow \quad y = -\dfrac{16}{x}$

Substitute: $x + 4 \times \left(-\dfrac{16}{x}\right) = 12$

Multiply by x: $x^2 - 64 = 12x \quad \Rightarrow \quad x^2 - 12x - 64 = 0$
$(x - 16)(x + 4) = 0 \quad \Rightarrow \quad x = -4$ or $x = 16$

When $x = -4$, $y = -\dfrac{16}{-4} = 4$

When $x = 16$, $y = -\dfrac{16}{16} = -1$

Check in $x + 4y = 12$: $-4 + 4 \times 4 = 12$ True

$16 + 4 \times (-1) = 12$ True

E.g. 5 Find the square root of $5 + 12\text{i}$.

Working Let $z = a + b\text{i}$ be such that $z = \sqrt{5 + 12\text{i}}$, where a and b are real

i.e. $a + b\text{i} = \sqrt{5 + 12\text{i}}$

Squaring both sides gives $a^2 - b^2 + 2ab\text{i} = 5 + 12\text{i}$

Equating real and imaginary parts:

Re: $a^2 - b^2 = 5$

Im: $2ab = 12 \quad ab = 6 \quad b = \dfrac{6}{a}$

Substituting: $a^2 - \left(\dfrac{6}{a}\right)^2 = 5$

$a^4 - 36 = 5a^2$

$a^4 - 5a^2 - 36 = 0$

$(a^2 - 9)(a^2 + 4) = 0$

$a^2 = 9$ or $a^2 = -4$

$\therefore a = \pm 3$ since $a^2 = -4$ gives imaginary values

When $a = 3, b = 2$

When $a = -3, b = -2$

The square roots of $5 + 12\text{i}$ are $3 + 2\text{i}$ and $-3 - 2\text{i}$.

E.g. 6 Find the square root of $3 - 4\text{i}$.

Working Let $z = a + b\text{i}$ be such that $z = \sqrt{3 - 4\text{i}}$, where a and b are real

i.e. $a + b\text{i} = \sqrt{3 - 4\text{i}}$

Squaring both sides gives $a^2 - b^2 + 2ab\text{i} = 3 - 4\text{i}$

Equating real and imaginary parts:

Re: $a^2 - b^2 = 3$

Im: $2ab = -4 \quad ab = -2 \quad b = -\dfrac{2}{a}$

Substituting: $a^2 - \left(-\dfrac{2}{a}\right)^2 = 3$

$a^4 - 4 = 3a^2$

$a^4 - 3a^2 - 4 = 0$

$(a^2 + 1)(a^2 - 4) = 0$

$a^2 = -1$ or $a^2 = 4$

$a = \pm 2$ since $a^2 = -1$ gives imaginary values

When $a = 2, b = -1$

When $a = -2, b = 1$

The square roots of $3 - 4\text{i}$ are $2 - \text{i}$ and $-2 + \text{i}$.

Video: Real and imaginary numbers
Video: Complex number arithmetic
Video: Square rooting complex numbers

Solutions to Starter and E.g.s

Exercise
p113 4A Qu 1i, 2i, 3i, 4i, 5i, 6i, 7i, 8i, 9i, 10, 13-17 (not done matrices yet)