Invariant points and invariant lines

Starter

1. A square undergoes a shear, $x-$axis invariant, mapping $(0,\ 1) \to (-4,\ 1)$. The point $(6,\ 2)$ is a vertex of the square before the shear. Find the new coordinates of the vertex.

Working: $x-$axis invariant $\Rightarrow$ $y-$coordinate is unchanged

$(0,\ 1) \to (-4,\ 1)$ $\Rightarrow$ the factor is -4

$$\begin{pmatrix}6\\2\end{pmatrix} \to \begin{pmatrix}6+(-4)\times 2\\2\end{pmatrix} = \begin{pmatrix}-2\\2\end{pmatrix}$$

The new coordinates of the vertex are $(-2,\ 2)$

2. Find the matrix which transforms $\begin{pmatrix}4\\3\end{pmatrix} \to \begin{pmatrix}9\\10\end{pmatrix}$ and $\begin{pmatrix}2\\1\end{pmatrix} \to \begin{pmatrix}5\\6\end{pmatrix}$.

Working: $\mathbf{M}\begin{pmatrix}4\\3\end{pmatrix} = \begin{pmatrix}9\\10\end{pmatrix}$ and $\mathbf{M}\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}5\\6\end{pmatrix}$

$$\therefore \mathbf{M}\begin{pmatrix}4&2\\3&1\end{pmatrix} = \begin{pmatrix}9&5\\10&6\end{pmatrix}$$

Post-multiply by $\begin{pmatrix}4&2\\3&1\end{pmatrix}^{-1}$: $\quad \mathbf{M} = \begin{pmatrix}9&5\\10&6\end{pmatrix}\begin{pmatrix}4&2\\3&1\end{pmatrix}^{-1}$

$$= \begin{pmatrix}3&-1\\4&-2\end{pmatrix}$$

3. Find the values of x and y such that:

(a) $\begin{pmatrix}2&3\\4&-1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix}$ (b) $\begin{pmatrix}-1&2\\1&0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix}$

Working: (a) $2x + 3y = x \quad \Rightarrow \quad x = -3y$

$4x - y = y \quad \Rightarrow \quad 2x = y$

Substituting gives $x = -6x$

This is only true when $x = 0$

If $x = 0$ then $y = 0$

The values that satisfy $\begin{pmatrix}2&3\\4&-1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix}$ are $x = 0, y = 0$.

(b) $-x + 2y = x \quad \Rightarrow \quad y = x$

$x = y$

So any point on the line $y = x$ satisfies $\begin{pmatrix}-1&2\\1&0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix}$.

E.g. 1 By letting $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, form and solve the simultaneous equations of $\mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ to show that $y = \left(\dfrac{a-c-1}{d-b-1}\right)x$ is the equation of invariant points.

Working:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$ax + by = x \quad \Rightarrow \quad (a-1)x + by = 0$

$cx + dy = y \quad \Rightarrow \quad cx + (d-1)y = 0$

Equate: $(a-1)x + by = cx + (d-1)y$

Rearrange: $(d-b-1)y = (a-c-1)x$

So $y = \left(\dfrac{a-c-1}{d-b-1}\right)x$ is the equation of a line of invariant points

E.g. 2 Using the equations formed from $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$, state what happens when:

(a) $a = 1$ (b) $d = 1$

Working:

(a) $ax + by = x$

When $a = 1$: $x + by = x \quad \Rightarrow \quad y = 0$

Substituting $y = 0$ in $cx + dy = y$: $x = 0$

So the origin is the unique invariant point.

(b) $cx + dy = y$

When $d = 1$: $cx + y = y \quad \Rightarrow \quad x = 0$

Substituting $x = 0$ in $ax + by = x$: $y = 0$

So the origin is unique invariant point

E.g. 3 Find the invariant points under the transformation $\begin{pmatrix} 4 & 1 \\ 6 & 3 \end{pmatrix}$.

Working:

$$\begin{pmatrix} 4 & 1 \\ 6 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$4x + y = x \quad \Rightarrow \quad y = -3x$

$6x + 3y = y \quad \Rightarrow \quad y = -3x$

The equations are equal so the invariant points under the transformation $\begin{pmatrix} 4 & 1 \\ 6 & 3 \end{pmatrix}$ are all the points lying on the line $y = -3x$ i.e. all points of the form $(k, -3k)$.

E.g. 4 Find the invariant points under the transformation $\begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$.

Working: $\begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$3x + 4y = x \quad \Rightarrow \quad x = -2y$

$x + 2y = y \quad \Rightarrow \quad x = -y$

Substituting gives $\quad y = 2y$

This is only true when $y = 0 \quad \Rightarrow \quad x = 0$

The invariant point under the transformation $\begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$ is $(0, 0)$.

E.g. 5 Find the equation of any invariant lines through the origin of the transformation whose matrix is $\begin{pmatrix} 2 & 3 \\ 0 & -1 \end{pmatrix}$.

Working: Any point on the invariant line has coordinates of the form (k, mk)

$\begin{pmatrix} 2 & 3 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} k \\ mk \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$x = 2k + 3mk \quad \Rightarrow \quad x = k(2 + 3m) \quad \Rightarrow \quad k = \dfrac{x}{2 + 3m}$

$y = -mk$

Substituting $k = \dfrac{x}{2 + 3m}$: $\quad y = \left(\dfrac{-m}{2 + 3m}\right)x$

Since the gradient must be the same as $y = mx$: $\quad m = \dfrac{-m}{2 + 3m}$

$3m^2 + 3m = 0 \quad \Rightarrow \quad m(m + 1) = 0 \quad \Rightarrow \quad m = 0$ & $m = -1$

So $y = 0$ and $y = -x$ are the invariant lines passing through the origin.

E.g. 6 Find the invariant lines of the matrix $\begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$ which pass through the origin.

Working: Any point on the invariant line has coordinates of the form (k, mk)

$\begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} k \\ mk \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$x = 3k + mk \quad \Rightarrow \quad x = k(3 + m) \quad \Rightarrow \quad k = \dfrac{x}{3 + m}$

$y = 2k + 4mk \quad \Rightarrow \quad y = k(2 + 4m)$

Substituting $k = \dfrac{x}{3 + m}$: $\quad y = \left(\dfrac{2 + 4m}{3 + m}\right)x$

Since the gradient must be the same as $y = mx$: $\quad m = \dfrac{2 + 4m}{3 + m}$

$m^2 - m - 2 = 0 \Rightarrow (m + 1)(m - 2) = 0 \quad \Rightarrow \quad m = -1$ & $m = 2$

So $y = -x$ and $y = 2x$ are the invariant lines passing through the origin.

Invariant lines not passing through the origin

An invariant line through the origin has the form $y = mx + c$.
Any point on this line is of the form $(k, mk + c)$

So if $y = mx + c$ is an invariant line: $\mathbf{M}\begin{pmatrix} k \\ mk + c \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

Again expand the matrix and form equations involving x and y.
By rearranging, k can be eliminated to form an equation where $y = f(m)x + g(m)c$.
The lines $y = mx + c$ and $y = f(m)x + g(m)c$ must be the same.
By equating coefficients of x and y we get: $m = f(m)$ and $1 = g(m)$
Solve the equation $m = f(m)$ to find values of m.
Check whether these $m-$values satisfy $1 = g(m)$:

If they do, $y = mx + c$ is an invariant line
If not, $1 = g(m)$ is only satisfied when $c = 0$ so $y = mx$ is an invariant line.

E.g. 7 Find the invariant lines of the matrix $\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$.

Working: Any point on the invariant line is of the form $(k, mk + c)$.

$$\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} k \\ mk + c \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$x = 2k + mk + c \quad \Rightarrow \quad x = k(2 + m) + c$$

$$\Rightarrow \quad k = \frac{1}{2 + m}x - \frac{1}{2 + m}c$$

$$y = 2k + 3mk + 3c \quad \Rightarrow \quad y = k(2 + 3m) + 3c$$

Substituting $k = \frac{1}{2 + m}x - \frac{1}{2 + m}c$:

$$y = \left(\frac{1}{2 + m}x - \frac{1}{2 + m}c\right)(2 + 3m) + 3c$$

$$y = \left(\frac{2 + 3m}{2 + m}\right)x + \left(3 - \frac{2 + 3m}{2 + m}\right)c$$

This image point must lie on the line $y = mx + c$, so equating coefficients of x and c:

$$x: \quad m = \frac{2 + 3m}{2 + m} \quad \Rightarrow \quad m^2 - m - 2 = 0$$

$$\Rightarrow \quad (m - 2)(m + 1) = 0$$

$$\Rightarrow \quad m = 2 \text{ or } m = -1$$

Now substitute into the equation formed by equating coefficients of c to see if it satisfies the equation.

$$c: \quad 3 - \frac{2 + 3m}{2 + m} = 1$$

$$m = 2: \quad 3 - \frac{2 + 3 \times 2}{2 + 2} = 3 - \frac{8}{4} = 1 \quad ✓$$

So $y = 2x + c$ is an invariant line.

$$m = -1: \quad 3 - \frac{2 + 3 \times (-1)}{2 + (-1)} = 3 - \frac{-1}{1} \neq 1$$

$\therefore \left(3 - \frac{2 + 3m}{2 + m}\right)c = c$ is only true when $c = 0$.

So $y = -x$ is the other invariant line.

E.g. 8 Find any invariant lines of the matrix $\begin{pmatrix} 4 & 3 \\ -3 & -2 \end{pmatrix}$

Working: Any point on the invariant line is of the form $(k, mk + c)$.

$$\begin{pmatrix} 4 & 3 \\ -3 & -2 \end{pmatrix}\begin{pmatrix} k \\ mk + c \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$x = 4k + 3mk + 3c \quad \Rightarrow \quad x = k(4 + 3m) + 3c$$

$$\Rightarrow \quad k = \frac{1}{4 + 3m}x - \frac{3}{4 + 3m}c$$

$$y = -3k - 2mk - 2c \quad \Rightarrow \quad y = k(-3 - 2m) - 2c$$

Substituting $k = \frac{1}{4 + 3m}x - \frac{3}{4 + 3m}c$:

$$y = \left(\frac{1}{4 + 3m}x - \frac{3}{4 + 3m}c\right)(-3 - 2m) - 2c$$

$$y = \left(\frac{-3 - 2m}{4 + 3m}\right)x + \left(\frac{3(3 + 2m)}{4 + 3m} - 2\right)c$$

This image point must lie on the line $y = mx + c$, so equating coefficients of x and c:

$$x: \quad m = \frac{-3 - 2m}{4 + 3m} \quad \Rightarrow \quad 3m^2 + 6m + 3 = 0$$

$$\Rightarrow \quad 3(m + 1)(m + 1) = 0$$

$$\Rightarrow \quad m = -1$$

Now substitute into the equation formed by equating coefficients of c to see if it satisfies the equation.

$$c: \quad \frac{3(3 + 2m)}{4 + 3m} - 2 = 1$$

$$m = -1: \quad \frac{3(3 + 2m)}{4 + 3m} - 2 = \frac{3(3 - 2)}{4 - 3} - 2 = 1 \quad ✓$$

So $y = -x + c$ is the unique invariant line.

Video A: Invariant points and lines
Video B: Invariant points and lines

Solutions to Starter and E.g.s

Exercise
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