Inverse of 2 by 2 Matrices

Starter

1. Find the values of k such that the matrix $\mathbf{M}$ is singular, where $\mathbf{M} = \begin{pmatrix} 2k & 18 \\ 4 & k-3.5 \end{pmatrix}$.

Working: det $\mathbf{M} = \left| \begin{pmatrix} 2k & 18 \\ 4 & k-3.5 \end{pmatrix} \right|$

$= 2k \times (k-3.5) - 4 \times 18$

$= 2k^2 - 7k - 72$

If $\mathbf{M}$ is singular, det $\mathbf{M} = 0$:

$2k^2 - 7k - 72 = 0$

$(2k+9)(k-8) = 0$

$k = -\dfrac{9}{2}$ or $k = 8$

2. The 2 by 2 identity matrix, $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Find:

(a) $\begin{pmatrix} 5 & -8 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Working:

(a) $\begin{pmatrix} 5 & -8 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & -8 \\ 3 & 4 \end{pmatrix}$

(b) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

3. Let matrix $\mathbf{X} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

(a) Find the values of a, b, c and d such that $\mathbf{X}\begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(b) Calculate the product $\begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix}\mathbf{X}$ using the values found in (a).

Working:

(a) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$a + 3b = 1 \quad 2a + 7b = 0 \quad \Rightarrow \quad a = 7 \;\&\; b = -2$

$c + 3d = 0 \quad 2c + 7d = 1 \quad \Rightarrow \quad c = -3 \;\&\; d = 1$

$\therefore \mathbf{X} = \begin{pmatrix} 7 & -2 \\ -3 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix}\mathbf{X} = \begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix}\begin{pmatrix} 7 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$;

4. Let matrix $\mathbf{P} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Find the values of a, b, c and d such that $\mathbf{P}\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Working: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$a + 3b = 1 \quad 2a + 6b = 0 \quad \Rightarrow$ No solution

$c + 3d = 0 \quad 2c + 6d = 1 \quad \Rightarrow$ No solution

The matrix $\mathbf{P}$ does not exist.

E.g. 1 Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and let $\mathbf{B} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

(a) Find the product $\mathbf{AB}$.

(b) State which needs to be done to $\mathbf{AB}$ make it equal to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(c) Hence state the inverse matrix of $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Include det $\mathbf{A}$ in your answer.

Working:

(a) $\mathbf{AB} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix}$

(b) Divide each element by $ad - bc$.

(c) $\mathbf{A}^{-1} = \dfrac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \dfrac{1}{\text{det } \mathbf{A}}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

E.g. 2 If it exists, find the inverse of the matrix:

(a) $\begin{pmatrix} 4 & 9 \\ 3 & 7 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & -6 \\ -3 & 9 \end{pmatrix}$ (c) $\begin{pmatrix} 5 & -3 \\ 10 & -5 \end{pmatrix}$

Working:

(a) $\left|\begin{pmatrix} 4 & 9 \\ 3 & 7 \end{pmatrix}\right| = 4 \times 7 - 3 \times 9 = 1$

$\begin{pmatrix} 4 & 9 \\ 3 & 7 \end{pmatrix}^{-1} = \dfrac{1}{1}\begin{pmatrix} 7 & -9 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} 7 & -9 \\ -3 & 4 \end{pmatrix}$

(b) $\left|\begin{pmatrix} 2 & -6 \\ -3 & 9 \end{pmatrix}\right| = 18 - 18 = 0$

No inverse

(c) $\left|\begin{pmatrix} 5 & -3 \\ 10 & -5 \end{pmatrix}\right| = -25 + 30 = 5$

$\begin{pmatrix} 5 & -3 \\ 10 & -5 \end{pmatrix}^{-1} = \dfrac{1}{5}\begin{pmatrix} -5 & 3 \\ -10 & 5 \end{pmatrix} = \begin{pmatrix} -1 & \frac{3}{5} \\ -2 & 1 \end{pmatrix}$

E.g. 3 Let $\mathbf{X} = \begin{pmatrix} 4 & 7 \\ 2 & 3 \end{pmatrix}$ and $\mathbf{Y} = \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}$. Calculate the inverse of $\mathbf{YX}$.

Working:

$$(\mathbf{YX})^{-1} = (\mathbf{X})^{-1}(\mathbf{Y})^{-1}$$

$$\mathbf{X}^{-1} = \frac{1}{-2}\begin{pmatrix} 3 & -7 \\ -2 & 4 \end{pmatrix}$$

$$\mathbf{Y}^{-1} = \frac{1}{-2}\begin{pmatrix} 6 & -4 \\ -5 & 3 \end{pmatrix}$$

$$(\mathbf{YX})^{-1} = \frac{1}{-2}\begin{pmatrix} 3 & -7 \\ -2 & 4 \end{pmatrix} \times \frac{1}{-2}\begin{pmatrix} 6 & -4 \\ -5 & 3 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 53 & -33 \\ -32 & 20 \end{pmatrix}$$

E.g. 4 Solve these matrix equations:

(a) $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}\mathbf{X} = \begin{pmatrix} -2 & 9 \\ -2 & 14 \end{pmatrix}$ (b) $\mathbf{X}\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} -2 & 9 \\ -2 & 14 \end{pmatrix}$

Hint: Remember to specify whether you are pre- or post-multiplying by a matrix.

Working:

(a) Pre-multiply by $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}^{-1}$:

$$\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}\mathbf{X} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}^{-1}\begin{pmatrix} -2 & 9 \\ -2 & 14 \end{pmatrix}$$

But $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \mathbf{I}$ and $\mathbf{IX} = \mathbf{X}$.

So $\mathbf{X} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}^{-1}\begin{pmatrix} -2 & 9 \\ -2 & 14 \end{pmatrix}$

$$\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}^{-1} = \frac{1}{-2}\begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix}$$

$$\therefore \mathbf{X} = \frac{1}{-2}\begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} -2 & 9 \\ -2 & 14 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix}$$

(b) Post-multiply by $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}^{-1}$:

$$\mathbf{X}\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} -2 & 9 \\ -2 & 14 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}^{-1}$$

But $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \mathbf{I}$ and $\mathbf{XI} = \mathbf{X}$.

So $\mathbf{X} = \begin{pmatrix} -2 & 9 \\ -2 & 14 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}^{-1}$

$$\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}^{-1} = \frac{1}{-2}\begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix}$$

$$\therefore \mathbf{X} = \begin{pmatrix} -2 & 9 \\ -2 & 14 \end{pmatrix} \times \frac{1}{-2}\begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 13 & -\frac{15}{2} \\ 18 & -10 \end{pmatrix}$$

Video: Inverse of 2 by 2 matrices
Video: Transposed and symmetric matrices

Inverse of 2 by 2 matrices EQ

Solutions to Starter and E.g.s

Exercise
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