Linear Simultaneous Equations

Starter

1. Find the inverse of the matrix $\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ -2 & -2 & 0 \end{pmatrix}$ by hand.

Working: $\begin{pmatrix} 1 & -1 & \frac{1}{2} \\ -1 & 1 & -1 \\ 1 & 0 & \frac{1}{2} \end{pmatrix}$

2. Using your calculator, solve for $\mathbf{X}$ the equation $\mathbf{X}\begin{pmatrix} 3 & -1 & 5 \\ 2 & 4 & -2 \\ 1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 18 & 29 & -3 \\ 7 & 0 & 12 \\ 7 & 0 & 10 \end{pmatrix}$.

Working: $\begin{pmatrix} 3 & -1 & 5 \\ 2 & 4 & -2 \\ 1 & 2 & 0 \end{pmatrix}^{-1} = \frac{1}{14}\begin{pmatrix} 4 & 10 & -18 \\ -1 & -5 & 16 \\ 0 & -7 & 14 \end{pmatrix}$

Post-multiply by $\begin{pmatrix} 3 & -1 & 5 \\ 2 & 4 & -2 \\ 1 & 2 & 0 \end{pmatrix}^{-1}$

$$\mathbf{X} = \begin{pmatrix} 18 & 29 & -3 \\ 7 & 0 & 12 \\ 7 & 0 & 10 \end{pmatrix}\frac{1}{14}\begin{pmatrix} 4 & 10 & -18 \\ -1 & -5 & 16 \\ 0 & -7 & 14 \end{pmatrix}$$

$$\mathbf{X} = \begin{pmatrix} 1 & 4 & 7 \\ 2 & -1 & 3 \\ 2 & 0 & 1 \end{pmatrix}$$

3. The system of simultaneous equations

$$2x + 3y = 1$$
$$4x - \ \ y = 9$$

can be expressed in matrix form $\begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 9 \end{pmatrix}$

Solve the equations by way of an inverse matrix.

Working: Pre-multiply by $\begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix}^{-1}$:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 9 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = -\frac{1}{14}\begin{pmatrix} -1 & -3 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 9 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

E.g. 1 Solve these equations using matrices:

(a) $x + 2y = 2, 3x + 4y = 8$ (b) $x + 2y = 1, 3x + 5y = 4$

Working: (a) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$

Pre-multiply by $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1}$: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1}\begin{pmatrix} 2 \\ 8 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 8 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

$x = 4, y = -1$

(b) $\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

Pre-multiply by $\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}^{-1}$: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 4 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = -\frac{1}{1}\begin{pmatrix} 5 & -2 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$x = 3, y = -1$

E.g. 2 Find the values of k for which these systems of equations have a unique solution:

(a) $x + 2y = 2, 3x + ky = 5$ (b) $3x - 4y = 7, kx - 3y = 8$

Working: (a) The matrix of coefficients is $\begin{pmatrix} 1 & 2 \\ 3 & k \end{pmatrix}$

For a unique solution $\begin{vmatrix} 1 & 2 \\ 3 & k \end{vmatrix} \neq 0$

$$k - 6 \neq 0$$

The equations have a unique solution when $k \neq 6$.

(b) The matrix of coefficients is $\begin{pmatrix} 1 & 2 \\ 3 & k \end{pmatrix}$

For a unique solution $\begin{vmatrix} 3 & -4 \\ k & -3 \end{vmatrix} \neq 0$

$$-9 + 4k \neq 0$$

The equations have a unique solution when $k \neq \frac{9}{4}$.

E.g. 3 Find the value of k for which these systems of equations do not have a unique solution. State whether there are no solutions or an infinite number of solutions and give a geometrical explanation.

(a) $x + y = 2, -x + ky = -2$ (b) $2x + 3y = 2, -6x + ky = -8$

Working: (a) The matrix of coefficients is $\begin{pmatrix} 1 & 1 \\ -1 & k \end{pmatrix}$.

For no unique solution $\begin{vmatrix} 1 & 1 \\ -1 & k \end{vmatrix} = 0$

$k + 1 = 0$

The equations have no unique solution when $k = -1$.

When $k = -1$ the equations are multiples of each other i.e. they are coincident (the same line) so there are infinite solutions.

(b) The matrix of coefficients is $\begin{pmatrix} 2 & 3 \\ -6 & k \end{pmatrix}$.

For no unique solution $\begin{vmatrix} 2 & 3 \\ -6 & k \end{vmatrix} = 0$

$2k + 18 = 0$

The equations have no unique solution when $k = -9$.

When $k = -9$ the coefficients are multiples of each other but the constants are not, so the lines are parallel. Hence there are no solutions, parallel.

E.g. 4 By using matrices, solve these systems of linear equations. You may use your calculator to find the inverse and perform matrix multiplication

(a) $2x - y - z = 4, x + 2y = 10, y - z = 2$

(b) $x + y - z = 3, 3x - 2y - z = 1, 2x + 3y - z = 9$

Working: (a) $\begin{pmatrix} 2 & -1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 10 \\ 2 \end{pmatrix}$

Pre-multiply by $\begin{pmatrix} 2 & -1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix}^{-1}$:

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 4 \\ 10 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix}$

$x = 4, y = 3, z = 1$

(b) $\begin{pmatrix} 1 & 1 & -1 \\ 3 & -2 & -1 \\ 2 & 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 9 \end{pmatrix}$

Pre-multiply by $\begin{pmatrix} 1 & 1 & -1 \\ 3 & -2 & -1 \\ 2 & 3 & -1 \end{pmatrix}^{-1}$:

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 3 & -2 & -1 \\ 2 & 3 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 3 \\ 1 \\ 9 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$

$x = 2, y = 2, z = 1$

Video: Solving simultaneous equations using matrices

Solutions to Starter and E.g.s

Exercise
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